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WEIERSTRASS SEMIGROUPS AT INFLECTION POINTS

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1. Introduction and Preliminaries

Let C be a smooth complex algebraic curve of genus g. For a divisor D on C, $\dim D$ means the dimension of the complete linear series $|D|$ containing D, which is the same as the projective dimension of the vector space of meromorphic functions f on C with divisor of poles $(f)_\infty \le D$.

Let $\mathcal{M}(C)$ denote the field of meromorphic functions on C. For points $p, q \in C$, we define the Weierstrass semigroup of a point and the Weierstrass semigroup of a pair of points by

$$H(p) = \{\alpha \in N \mid \text{ there exists } f \in \mathcal{M}(C) \text{ with } (f)_\infty = \alpha p\}$$
$$H(p,q) = \{(\alpha, \beta) \in N \times N \mid \text{there exists } f \in \mathcal{M}(C) \text{ with } (f)_\infty = \alpha p + \beta q\},$$

where N denotes the set of non-negative integers. Indeed, these sets form sub-semigroups of N and $N \times N$, respectively. We know that the cardinality of the Weierstrass gap sequence $G(p) = N \setminus H(p)$ is equal to the genus g of the given curve. An element of $H(p)$ [resp. $G(p)$] is called a nongap [resp. gap] at p. Recall that p is said to be a Weierstrass point if $G(p) \neq \{1, 2, \cdots, g\}$. Also recall that a point p is a Weierstrass point if and only if $\dim |gp| \ge 1$.

For a nonsingular plane curve C and a line L, we denote $C.L$ the divisor on C cut out by L. We know that if $C.L = dp$, where $d = \deg C \ge 4$, then p is a Weierstrass point whose Weierstrass semigroup is $\{i(d-1) + jd \mid i, j \in N\}$.**[CK]**

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In this paper, we consider a point p on C, of degree $d \geq 4$, at which the tangent line of C meets at only one other point with C. We can easily prove that if $C.L = mp + nq$, $m, n \in N, m > d/2$, then p is a Weierstrass point of C. More generally we obtain

THEOREM 1.1. *Let C be a nonsingular curve of degree $d \geq 4$. If $C.L \geq mp$ for $m > d/2$, then p is a Weierstrass point of C.*

Proof. Let $D = C.L = mp+q_1+q_2+\cdots+q_n$ where $m+n = d$ and q_i are points of C. Since the canonical series K are cut out by the system of degree $d-3$ curves, $K = |(d-3)D|$. Since $m(d-3) - \frac{d^2-3d+2}{2} \geq 0$, we have

$$\begin{aligned}
\dim |gp| &= \dim \left| \frac{d^2-3d+2}{2} p \right| \\
&= \dim \left| (d-3)D - (d-3)(q_1 + \cdots + q_n) - (m(d-3) - \frac{d^2-3d+2}{2})p \right| \\
&= \dim(K - (d-3)(q_1 + \cdots + q_n) - \left(m(d-3) - \frac{d^2-3d+2}{2} p\right) \\
&\geq g - 1 - n(d-3) - m(d-3) + g \\
&= 2g - 1 - (d-3)(m+n) = 1.
\end{aligned}$$

Thus p is a Weierstrass point of C.

In section 2, we determine completely the Weierstrass semigroups of p for $m = d-1,\ d-2$, especially. In section 3, we consider the Weierstrass semigroup of p for $m = d-3$.

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2. Weierstrass semigroups at inflection points of order $d-1$ and $d-2$

We denote C a smooth complex plane curve of degree d. Let p be a point on C and L be a tangent line of C at p. Assume that $C.L = mp+nq$, $m, n \in N$. Note that the genus of C is $(d-1)(d-2)/2$.

In this section, we find Weierstrass semigroups at inflection points for the cases $m = d-1$, $d-2$.

LEMMA 2.1. *Let* $(\alpha,\beta) \in H(p,q)$ *with* $\alpha \geq 1$. *If* $\dim|\gamma p+\beta q| = \dim|(\gamma-1)p+\beta q|+1$ *for* $\gamma \geq \alpha$, *then* $(\gamma,\beta) \in H(p,q)$.

Proof. See [K].

LEMMA 2.2. *For* $\alpha,\beta \geq 1$, $(\alpha,\beta) \in H(p,q)$ *if and only if*

$$\begin{aligned}\dim|\alpha p+\beta q| &= \dim|(\alpha-1)p+\beta q|+1\\ &= \dim|\alpha p+(\beta-1)q|+1.\end{aligned}$$

In other words, $(\alpha,\beta) \in H(p,q)$ *if and only if the linear series* $|\alpha p+\beta q|$ *is base point free.*

Proof. See [K].

LEMMA 2.3. *If* (α,β), $(\gamma,\delta) \in H(p,q)$ *and* $\alpha \geq \gamma$, $\beta \leq \delta$, *then* $(\alpha,\delta) \in H(p,q)$.

Proof. See [K].

THEOREM 2.4. *If* $d \geq 4$ *and* $C.L = (d-1)p+q$, *then the Weierstrass semigroup at* p *is*

$$\begin{aligned}&(d-1),\\ &2(d-1)-1,\ 2(d-1),\\ &3(d-1)-2,\ 3(d-1)-1,\ 3(d-1),\\ &\cdots\cdots,\\ &(d-3)(d-1)-(d-4),\ (d-3)(d-1)-(d-5),\ \longrightarrow,\ (d-3)(d-1),\\ &(d-3)(d-1)+2\ \longrightarrow,\end{aligned}$$

where $\longrightarrow$ *means the consecutive integers.*

Proof. Note that the series $|(d-1)p+q|$ cut out by lines and the series $|(d-1)p|$ [resp. $|(d-2)p+q|$] cut out by lines through q [resp. p] are base point free. Thus the pairs $(d-1,1)$, $(d-1,0)$ and $(d-2,1)$ are elements of $H(p,q)$. Since $H(p,q)$ is a semigroup,

$$\{(i(d-1)-k,j) \mid i \in N,\ j=0,1,2,\cdots,i,\ k=0,1,2,\cdots,j\} \subset H(p,q).$$

From definitions, we have $H(p)=\{\alpha \mid (\alpha,0) \in H(p,q)\}$, and hence $\{i(d-1) \mid i \in N\} \subset H(p)$. For $1 \leq i \leq d-2$, since the linear series

$|i(d-1)p+iq|$ is cut out by degree i curves, we have $\dim|i(d-1)p+iq| = i(i+3)/2$, and hence $\dim|i(d-1)p| = i(i+1)/2$ by Lemma 2.2. Thus there are exactly $i-1$ elements of $H(p)$ between $(i-1)(d-1)$ and $i(d-1)$. We claim that these elements are $i(d-1)-j$, $j=1,2,\cdots,i-1$. For this, we show that the numbers $i(d-1)-j$, $j=i,i+1,\cdots,d-2$ are not elements of $H(p)$. Suppose that $i(d-1)-j \in H(p)$, i.e., $(i(d-1)-j,0) \in H(p,q)$, for some $j=i,i+1,\cdots,d-2$. By Lemma 2.3, $(i(d-1)-j,i-1) \in H(p,q)$. But this contradicts the facts $\dim|(i-1)(d-1)p+(i-1)q| = (i-1)(i+2)/2$ and $\dim|(i(d-1)-(i-1))p+(i-1)q| = i(i+1)/2$. Note that the difference of these two numbers is one, hence there is no element of the form $(i(d-1)-j,i-1)$ for $j=i,i+1,\cdots,d-2$.

Thus we have proved the elements appeared in the statement are elements of $H(p)$.

Conversely, we prove that there are no elements of $H(p)$ other than those numbers. To do this, we count the natural numbers omitted in the sequence, and we get this number is just $g=(d-1)(d-2)/2$, the genus of the curve. Note that the cardinality of the complement of a Weierstrass semigroup is just g.

Thus the proof is complete.

THEOREM 2.5. *If $d \geq 5$ and $C.L = (d-2)p+2q$, then the Weierstrass semigroup at p is*

$2(d-2)$,

$3(d-2)-1,\ 3(d-2)$,

$4(d-2)-2,\ 4(d-2)-1,\ 4(d-2)$,

$\cdots\cdots$,

$(d-3)(d-2)-(d-5),\ (d-3)(d-2)-(d-6),\ \longrightarrow,\ (d-3)(d-2)$,

$(d-3)(d-2)+2\ \longrightarrow$.

Proof. As in the proof of Theorem 2.4, the series

$$|(d-2)p+2q|, \quad |(d-2)p+q| \quad \text{and} \quad |(d-3)p+2q|$$

are base point free, hence the pairs $(d-2,2)$, $(d-2,1)$ and $(d-3,2)$

are elements of $H(p,q)$. Since $H(p,q)$ is a semigroup,

$$\{(i(d-2)-k, j+i) \mid i \in N,\ j=0,1,2,\cdots,i,\quad k=0,1,2,\cdots,j\} \subseteq H(p,q).$$

At first, we show that $2(d-2)$ is the first nongap at p. Since the linear series $|2(d-2)p+4q|$ cut out by conics is of dimension 5, we have $\dim|2(d-2)p| \geq 1$. On the other hand, for $0 < \alpha \leq d-2$, α is not an element of $H(p)$, because $\dim|(d-2)p+q| = 1$ and $(d-2,1) \in H(p,q)$. Moreover, since $\dim|(d-2)p+2q|+1 = \dim|2(d-2)p+2q| = 3$, there can not exist a nongap between $d-2$ and $2(d-2)$ by Lemma 2.3. Thus we conclude that $2(d-2)$ is the first nongap at p.

For each $3 \leq i \leq d-2$, we show that there does not exist a nongap between $(i-1)(d-2)$ and $i(d-2)-(i-2)$. To do this, we compute the dimensions of two linear series $|(i-1)(d-2)p+2(i-1)q|$ and $|(i(d-2)-(i-2))p+2(i-1)q|$. Since the series

$$|(i-1)(d-2)p+2(i-1)q|$$

is cut out by the system of degree $i-1$ curves, its dimension is $(i-1)(i+2)/2$. And, since

$$|(i(d-2)-(i-2))p+2(i-1)q| = |i(d-2)p+2iq|-((i-2)p+2q),$$

$\dim|(i(d-2)-(i-2))p+2(i-1)q| = i(i+3)/2 - i = (i-1)(i+2)/2+1$ by Lemma 2.2. Thus the claim follows from Lemma 2.3.

Counting the number of gaps appeared in above 2 steps, we see that it is just g. Hence the remainder of them form a Weierstrass semigroup at p. Thus the proof is complete.

Remark 2.6. Since it is easy to construc the curves in Theorem 2.4 and Theorem 2.5, such semigroups are Weierstrass semigroups.

3. Weierstrass semigroups at inflection points of order $d-3$

For the case $m = d-3$, the Weierstrass semigroups are more complicated than previous cases.

THEOREM 3.1. *If $d \geq 7$ and $C.L = (d-3)p + 3q$, then the Weierstrass semigroup at p is the subset of the set*

$$\begin{aligned}&\{3(d-3),\\ &4(d-3)-1,\ 4(d-3),\\ &5(d-3)-2,\ 5(d-3)-1,\ 5(d-3),\\ &\cdots\cdots,\\ &(d-3)(d-3)-(d-6),\ (d-3)(d-3)-(d-7),\ \longrightarrow,\ (d-3)(d-3),\\ &(d-3)(d-3)+2\ \longrightarrow\}.\end{aligned}$$

Proof. As in the proof of Theorem 2.4, the series

$$|(d-3)p+3q|,\quad |(d-3)p+2q| \text{ and } |(d-4)p+3q|$$

are base point free, hence the pairs $(d-3,3)$, $(d-3,2)$ and $(d-4,3)$ are elements of $H(p,q)$. Since $H(p,q)$ is a semigroup,

$$\{(i(d-3)-k, j+2i) \mid i \in N,\ j = 0,1,2,\cdots,i,\quad k = 0,1,2,\cdots,j\} \\ \in\ H(p,q).$$

First we show that there is no nonzero nongap less than $3(d-3)$. Since the series $|3(d-3)p+9q|$ cut out by cubics has dimension 9, and since $(3(d-3),\alpha) \in H(p,q)$ for $2 \leq \alpha \leq 9$ by below Lemma 3.2, we have $\dim|3(d-3)p+2q| = 2$, which is same as $\dim|(d-3)p+2q|+1$. Thus there does not exist a nongap between $d-3$ and $3(d-3)$ by Lemma 2.3. And since $\dim|(d-3)p+2q| = 1$, there is no nonzero nongap less than or equal to $d-3$.

For each $4 \leq i \leq d-2$, we show that there does not exist a nongap between $(i-1)(d-3)$ and $i(d-3)-(i-3)$. To do this, we compute the dimensions of two linear series. Since the series $|(i-1)(d-3)p+3(i-1)q|$ is cut out by the system of degree $i-1$ curves, its dimension is $(i-1)(i+2)/2$. And, since

$$|(i(d-3)-(i-3))p+3(i-1)q| = |i(d-3)p+3iq| - ((i-3)p+3q),$$

$\dim|(i(d-3)-(i-3))p+3(i-1)q| = i(i+3)/2 - i = (i-1)(i+2)/2+1$ by Lemma 2.2. Thus the claim follows from Lemma 2.3.

LEMMA 3.2. *Under the same assumption as Theorem 3.1,* $(3(d-3), j) \in H(p,q)$ *for* $2 \leq j \leq 9$.

Proof. From the first part of the proof of Theorem 3.1, we know that $(3(d-3), j) \in H(p,q)$ for $6 \leq j \leq 9$. By Lemma 2.2 we have

$$\dim |3(d-3)p + 5q| = 5.$$

Since $\dim |2(d-3)p + 5q| = 4$, using Lemma 2.1, there exists an element of $H(p,q)$ which is of the form $(\alpha, 5)$ with $2(d-2) < \alpha \leq 3(d-3)$. If $\alpha \neq 3(d-3)$, then the element $(\alpha, 6)$ is also an element in $H(p,q)$ by Lemma 2.3, and hence we have $\dim |3(d-3)p + 6q| > 6$ by Lemma 2.2. This contradicts the fact that $\dim |3(d-3)p + 6q| = 6$.

Repeating similar process as above, we obtain the result.

REMARK 3.3. Note that the number of elements of the complement of the set in Theorem 3.2 is $g-1$. Thus we know that exactly one gap is contained in that set.

REMARK 3.4. Under the same assumption as Theorem 3.1, there is no canonical divisor containing the divisor $((d-3)(d-3)+1)p$ if and only if $(d-3)(d-3)+2 \longrightarrow$ are nongaps at p. In this case, we conclude that one element of the set $\{3(d-3),\ 4(d-3)-1, 4(d-3),\ 5(d-3)-2, 5(d-3)-1, 5(d-3),\ \cdots,\ (d-3)(d-3)-(d-6), (d-3)(d-3)-(d-7) \longrightarrow (d-3)(d-3)\}$ is a gap at p.

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