J. Korean Math. Soc. **32** (1995), No. 4, pp. 725–737

A CENTRAL LIMIT THEOREM FOR SOJOURN TIME OF STRONGLY DEPENDENT 2-DIMENSIONAL GAUSSIAN PROCESS

TAE IL JEON

1. Introduction

Let $\boldsymbol{X}_t = (X_t^{(1)}, X_t^{(2)})', t \geqslant 0$, be a real stationary 2-dimensional Gaussian process with $EX_t^{(1)} = EX_t^{(2)} = 0$ and

$$E\boldsymbol{X}_0\boldsymbol{X}_t' = \begin{pmatrix} r(t) & \rho(t) \\ \rho(t) & r(t) \end{pmatrix},$$

where $r(t) \sim |t|^{-\alpha}, 0 < \alpha < 1/2, \rho(t) = o(r(t))$ as $t \to \infty, r(0) = 1$, and $\rho(0) = \rho$ $(0 \leqslant \rho < 1)$. For $t > 0, u > 0$, and $v > 0$, let $L_t(u, v)$ be the time spent by $\boldsymbol{X}_s, 0 \leqslant s \leqslant t$, above the level (u, v). In other words $L_t(u, v)$ be the time spent by $\boldsymbol{X}_s$ in the region $\{(x, y) | \ x > u, \ y > v\}$. Let $u = u(t), v = v(t)$ be increasing functions at a sufficiently slow rate. Note that $r(t)$ is nonnegative for all t and tends to 0 as $|t|$ approaches infinity. If the assumptions stated above hold, then $L_t(u, v)$, upon appropriate normalization, has a limiting normal distribution.

For the case of one-dimensional process X_t with a sufficiently slow correlation function $r(t)$, Berman [1] and Maejima [6,and 7] have formulated central limit theorems for the high level sojourn. For given multi-dimensional stationary Gaussian processes with long-range dependence, Berman [2] and Maejima [8] have treated the limiting behavior of the time spent by those processes in some particular domains in multi-dimensional spaces. They have formulated non-central limit theorems which have been characterized in terms of Rosenblatt distribution.

Received September 17, 1994.
1991 AMS Subject Classification: Primary 60F05, secondary 60G15.
Key words: central limit theorem, sojourn, diagram formula.
Research supported by The KOSEF 1993.

Many authors [3,4,5,9, and 10] have investigated the limiting behavior of non-linear functionals of stationary Gaussian processes with long-range dependence. The functionals used in the previously mentioned papers are more general than sojourn functionals even they should satisfy some other conditions. Not many papers appeared in the study of limiting distributions for 2-dimensional processes because of the complexity of computation caused by the effect of cross correlation functions when we allow the long-range dependence in the process.

The basic idea in the proof of the main result of this paper is similar to that in the early papers [1 and 3]. But the conditions of correlation functions are quite different from that in those papers. The diagram formula is used in computation of expectations for the product of Hermite polynomials of vector process.

2. Results

Let $f(x,y)$ be a real valued function such that

$$\begin{aligned} &\int_{\infty}^{\infty}\int_{\infty}^{\infty} f(x,y)\phi(x,y)\,dx\,dy = 0, \\ (1) \qquad &\int_{\infty}^{\infty}\int_{\infty}^{\infty} f(x,y)^2\phi(x,y)\,dx\,dy < \infty, \end{aligned}$$

where $\phi(x,y)$ is the bivariate standard normal density function. Then, as is well known, f has an expansion with respect to the Hermite polynomials with leading coefficients 1. Hermite polynomials of two variables are defined by

$$H_{\boldsymbol{m}}(x,y) = (-1)^{|\boldsymbol{m}|}\exp\Big(\frac{x^2+y^2}{2}\Big)\frac{\partial^{|\boldsymbol{m}|}}{\partial x^{m_1}\partial y^{m_2}}\exp\Big(-\frac{x^2+y^2}{2}\Big),$$

where $\boldsymbol{m} = (m_1, m_2)$, and $|\boldsymbol{m}| = m_1 + m_2$.

By the argument above, f has the expression

$$f(x,y) = \sum_{j=0}^{\infty}\Big(\sum_{|\boldsymbol{m}|=j} c_{\boldsymbol{m}} H_{\boldsymbol{m}}(x,y)\Big),$$

with

$$c_{\boldsymbol{m}} = \int_{\infty}^{\infty} \int_{\infty}^{\infty} f(x,y)\phi(x,y)H_{\boldsymbol{m}}(x,y)dxdy.$$

Let $\boldsymbol{X}_t = (X_t^{(1)}, X_t^{(2)}), t \geqslant 0$, be a stationary vector Gaussian process with $EX_t^{(1)} = EX_t^{(2)} = 0$ and $EX_t^{(i)}X_{t+s}^{(i)} = r(s), \quad i = 1, 2, EX_t^{(i)} X_{t+s}^{(j)} = \rho(s), (i,j) = (1,2)$ or $(2,1)$ for all t, n. Without loss of generality we may assume $r(0) = 1, \rho(0) = 0$.

Let $L_t(u,v)$ be the time spent by $\boldsymbol{X}_s, 0 \leqslant s \leqslant t$, above the level (u,v), which means the domain $\{(x,y) \in \boldsymbol{R}^2 | \ x > u, \ y > v\}$. Then $L_t(u,v)$ has the following expression

$$L_t(u,v) = \int_0^t I[X_s^{(1)} > u, X_s^{(2)} > v]ds,$$

where I is the indicator function. As is customary, we write

$$\int_u^{\infty} \phi(x)dx = 1 - \Phi(u),$$

where $\phi(x)$ is the standard normal density function. Since $f(x,y) = I[x > u, y > v]$ satisfies (1) it has the Hermite expansion

$$\begin{aligned} f(x,y) =& [1 - \Phi(u)][1 - \Phi(v)] \\ &+ \{\phi(u)[1 - \Phi(v)]H_1(x) + [1 - \Phi(u)]\phi(v)H_1(y)\} \\ &+ \sum_{j=2}^{\infty} \sum_{|\boldsymbol{m}|=j} B(m_1,u)B(m_2,v)H_{m_1}(x)H_{m_2}(y), \end{aligned} \tag{2}$$

where

$$B(k,w) = \begin{cases} 1 - \Phi(w), & \text{if } k = 0 \\ \frac{1}{k!}H_{k-1}(w)\phi(w) & \text{if } 1 \leqslant k \leqslant j. \end{cases} \tag{3}$$

It follows from (2) that

(4)

$$\begin{aligned} L_t(u,v) =& \int_0^t f(\boldsymbol{X}_s)ds = [1 - \Phi(u)][1 - \Phi(v)]t \\ &+ \phi(u)[1 - \Phi(v)] \int_0^t H_1(X_s^{(1)})ds + [1 - \Phi(u)]\phi(v) \int_0^t H_1(X_s^{(2)})ds \\ &+ \sum_{j=2}^{\infty} \sum_{|\boldsymbol{m}|=j} B(m_1,u)B(m_2,v) \int_0^t H_{m_1}(X_s^{(1)})H_{m_2}(X_s^{(2)})ds. \end{aligned}$$

We introduce a diagram formula. In order to set up the formula we need some preliminaries. We call an undirected graph G with $\sum_{j=1}^{p}(l_1^j + l_2^j)$ vertices a diagram of order $\boldsymbol{l} = (\boldsymbol{l}^1, \boldsymbol{l}^2, \dots, \boldsymbol{l}^p)$, where $\boldsymbol{l}^j = (l_1^j, l_2^j)$ for $j = 1, \dots, p$, if it satisfies the following three conditions:

[i] The set of vertices V of the graph G has the form

$$V = \bigcup_{j=1}^{p} S_j,$$

where

$$\begin{gathered} S_j = L_1^j \cup L_2^j, \\ L_1^j = \{(2j-1, n) \mid 1 \leqslant n \leqslant l_1^j\}, \\ L_2^j = \{(2j, n) \mid 1 \leqslant n \leqslant l_2^j\}, \\ j = 1, \dots, p. \end{gathered}$$

(define $L_k^j = \emptyset$ for $l_k^j = 0, k = 1, 2$) We call S_j the *sector* j of the graph G, L_1^j the *level* $(2j-1)$ of the graph G, and L_2^j the *level* $(2j)$ of the graph G.

[ii] Every vertex is of degree 1.

[iii] Edges may pass only between different sectors, in other words, no edge passes between levels L_1^j and L_2^j, for $j = 1, \dots, p$.

Let $\Gamma = \Gamma(\boldsymbol{l}) = \Gamma(\boldsymbol{l}^1, \dots, \boldsymbol{l}^p)$ denote the set of diagrams with properties [i],[ii] and [iii] above. Given a graph $G \in \Gamma$ let $V(G)$ be the set of all edges of G and let $V_G(L_{i_1}^{j_1}, L_{i_2}^{j_2}), i_1, i_2 = 1, 2;\ j_1, j_2 = 1, \dots, p$ be the set of all edges pass between levels $L_{i_1}^{j_1}$ and $L_{i_2}^{j_2}$. If $\omega \in V_G(L_{i_1}^{j_1}, L_{i_2}^{j_2})$, then define $d_1(\omega) = j_1$ and $d_2(\omega) = j_2$. Define $k_G(j) = |\{\omega \in G \mid d_1(\omega) = j\}|$. Indeed $k_G(j)$ is the cardinality of the set of edges $\omega \in V(G)$ which begin at the *sector* j and end at *sectors* of indices higher than j. We state the diagram formula for random vectors without proof. See [9] for the detail.

Lemma 1 (Diagram Formula). *Let* $(\boldsymbol{X}_{t_1}, \dots, \boldsymbol{X}_{t_p}), p \geqslant 2,$ *be such that* $\boldsymbol{X}_{t_i} = (X_{t_i}^{(1)}, X_{t_i}^{(2)}), i = 1, \dots, p$ *are jointly normal and for* each $t, s, \in \{t_1, \dots, t_p\}$, $EX_t^{(1)} = EX_t^{(2)} = 0$, $E(X_t^{(1)})^2 = E(X_t^{(2)})^2 = 1$, $EX_t^{(1)}X_t^{(2)} = 0$, $EX_t^{(i)}X_s^{(j)} = r_{ij}(s-t), i, j = 1, 2$. *Then we have*

$$E\Big[\prod_{i=1}^{p} H_{\boldsymbol{l}^i}(X_{t_i}^{(1)}, X_{t_i}^{(2)})\Big] = \sum_{G\in\Gamma(\boldsymbol{l})} \prod_{\omega\in V(G)} u(t_{d_1(\omega)} - t_{d_2(\omega)}),$$

where $u(t_{d_1(\omega)} - t_{d_2(\omega)}) = r_{i_1 i_2}(t_{j_1} - t_{j_2})$ *if* $\omega \in V_G(L_{i_1}^{j_1}, L_{i_2}^{j_2})$ *and* $\boldsymbol{l} = (\boldsymbol{l}^1, \dots, \boldsymbol{l}^p)$.

Note that the 2-dimensional Hermite polynomial of order (m_1, m_2) can be expressed as a product of one-dimensional Hermite polynomials of order m_1 and m_2;

$$H_{\boldsymbol{m}}(x, y) = H_{(m_1, m_2)}(x, y) = H_{m_1}(x)H_{m_2}(y).$$

As a special case of the diagram formula, $p = 2$, we have

$$E\left[H_{\boldsymbol{m}}(X_t^{(1)}, X_t^{(2)})H_{\boldsymbol{n}}(X_s^{(1)}, X_s^{(2)})\right] = 0, \tag{5}$$

if $|\boldsymbol{m}| \neq |\boldsymbol{n}|$, because there is no diagram between two sectors with different numbers of vertices. The property (11) is called the orthogonality of Hermite polynomials. By Lemma 1 we have

$$E\Big\{\int_0^t H_{m_1}(X_s^{(1)})H_{m_2}(X_s^{(2)})ds\Big\} = 0,$$

$$E\Big\{\sum_{|\boldsymbol{m}|=j} \int_0^t H_{m_1}(X_s^{(1)})H_{m_2}(X_s^{(2)})\Big\}^2$$

$$= \sum_{|\boldsymbol{m}|=|\boldsymbol{n}|=j} \sum_{k=max(0,m_1-n_2)}^{min(m_1,n_1)} \frac{m_1!m_2!n_1!n_2!}{k!(n_1-k)!(m_1-k)!(n_2-m_1+k)!}$$
$$\times \int_0^t \int_0^t r^{n_2-m_1+2k}(s_1-s_2)\rho^{m_1+n_1-2k}(s_1-s_2)ds_1ds_2,$$

and

$$E\Big\{\int_0^t H_{m_1}(X_s^{(1)})H_{m_2}(X_s^{(2)})ds \int_0^t H_{n_1}(X_s^{(1)})H_{n_2}(X_s^{(2)})ds\Big\} = 0 \quad \text{if } |\boldsymbol{m}| \neq |\boldsymbol{n}|. \tag{6}$$

Since the Hermite polynomial $H_1(x) = x$, we may rewrite (4) by

$$\begin{aligned} &L_t(u,v) - [1-\Phi(u)][1-\Phi(v)]t \\ &= \phi(u)[1-\Phi(v)]\int_0^t X_s^{(1)}ds + [1-\Phi(u)]\phi(v)\int_0^t X_s^{(2)}ds \\ &+ \sum_{j=2}^{\infty}\sum_{|\boldsymbol{m}|=j} B(m_1,u)B(m_2,v)\int_0^t H_{m_1}(X_s^{(1)})H_{m_2}(X_s^{(2)})ds. \end{aligned} \tag{7}$$

Let us compute the variance of $L_t(u,v)$

$$\begin{aligned} Var[L_t(u,v)] = A_t(u,v) &+ \sum_{j=2}^{\infty}\sum_{|\boldsymbol{m}|=|\boldsymbol{n}|=j} B_0(\boldsymbol{m},\boldsymbol{n},u,v) \\ &\times \sum_{k=max(0,m_1-n_2)}^{min(m_1,n_1)} \frac{m_1!m_2!n_1!n_2!}{k!(n_1-k)!(m_1-k)!(n_2-m_1+k)!} \\ &\times \int_0^t\int_0^t r^{n_2-m_1+2k}(s_1-s_2)\rho^{m_1+n_1-2k}(s_1-s_2)ds_1ds_2, \end{aligned}$$

where

$$\begin{aligned} A_t(u,v) =2\Big\{&\phi^2(u)[1-\Phi(v)]^2\int_0^t (t-s)r(s)ds \\ &+ 2\phi(u)[1-\Phi(v)][1-\Phi(u)]\phi(v)\int_0^t (t-s)\rho(s)ds \\ &+ [1-\Phi(u)]^2\phi^2(v)\int_0^t (t-s)r(s)ds\Big\} \end{aligned}$$

and

$$B_0(\boldsymbol{m},\boldsymbol{n},u,v) = B(m_1,u)B(m_2,v)B(n_1,u)B(n_2,v).$$

Here $B(k,w)$ satisfies (3), for instance,

$$\begin{aligned}&B_0((0,j),(j,0),u,v)\\=&\frac{1}{(j!)^2}[1-\Phi(u)]H_{j-1}(v)\phi(v)H_{j-1}(u)\phi(u)[1-\Phi(v)].\end{aligned}$$

Let, for the simplicity,

(8)

$$A_t'(u,v)=\left\{[1-\Phi(u)]^2\phi^2(v)+\phi^2(u)[1-\Phi(v)]^2\right\}\int_0^t(t-s)r(s)ds.$$

Then we have

$$A_t(u,v)=2A_t'(u,v)+4\phi(u)[1-\Phi(v)][1-\Phi(u)]\phi(v)\int_0^t(t-s)\rho(s)ds.$$

Consider a new process

$$Y_t(u,v)=\frac{L_t(u,v)-[1-\Phi(u)][1-\Phi(v)]t}{A_t(u,v)^{1/2}}.$$

We formulate the following main result.

THEOREM 1. *Let* $\boldsymbol{X}_t=(X_t^{(1)},X_t^{(2)})',t\geqslant 0$, *be a real stationary 2-dimensional Gaussian process with* $EX_t^{(1)}=EX_t^{(2)}=0$ *and*

$$E\boldsymbol{X}_0\boldsymbol{X}_t'=\begin{pmatrix}r(t)&\rho(t)\\\rho(t)&r(t)\end{pmatrix},$$

where $r(t)\sim|t|^{-\alpha},0<\alpha<1/2,\rho(t)\le r(t),\rho(t)=o(r(t))$ *as* $|t|\to\infty,r(0)=1$, *and* $\rho(0)=\rho\ (0\leqslant\rho<1)$. *and* $u=u(t)\sim t^{\beta/2},v=v(t)\sim t^{\beta/2}$, *for* $0<\beta<\alpha$. *Let* $L_t(u,v)$ *be the time spent by* $\boldsymbol{X}_s,0\leqslant s\leqslant t$, *above the level* (u,v). *Then*

$$Y_t(u,v)\xrightarrow{d}\mathcal{N}(0,1).$$

where $\xrightarrow{d}$ *means convergence in distribution and* $\mathcal{N}(0,1)$ *is the standard normal random variable.*

Proof. Rewriting $Y_t(u,v)$, we have

$$Y_t(u,v) = \frac{\phi(u)[1-\Phi(v)]\int_0^t X_s^{(1)}ds + [1-\Phi(u)]\phi(v)\int_0^t X_s^{(2)}ds}{A_t(u,v)^{1/2}}$$
$$+\frac{\sum_{j=2}^{\infty}\sum_{|\boldsymbol{m}|=j} B(m_1,u)B(m_2,v)\int_0^t H_{m_1}(X_s^{(1)})H_{m_2}(X_s^{(2)})ds}{A_t(u,v)^{1/2}}.$$

And

$$\begin{aligned}
&Var(Y_t(u,v))\\
=\;& Var\left(\frac{\phi(u)[1-\Phi(v)]\int_0^t X_s^{(1)}ds + [1-\Phi(u)]\phi(v)\int_0^t X_s^{(2)}ds}{A_t(u,v)^{1/2}}\right)\\
&+Var\left(\frac{\sum_{j=2}^{\infty}\sum_{|\boldsymbol{m}|=j} B(m_1,u)B(m_2,v)\int_0^t H_{m_1}(X_s^{(1)})H_{m_2}(X_s^{(2)})ds}{A_t(u,v)^{1/2}}\right)\\
=\;& 1+Var\left(\frac{\sum_{j=2}^{\infty}\sum_{|\boldsymbol{m}|=j} B(m_1,u)B(m_2,v)\int_0^t H_{m_1}(X_s^{(1)})H_{m_2}(X_s^{(2)})ds}{A_t(u,v)^{1/2}}\right),
\end{aligned}$$

by the orthogonality conditon (6).

LEMMA 2.

$$Var\left(\frac{\sum_{j=2}^{\infty}\sum_{|\boldsymbol{m}|=j} B(m_1,u)B(m_2,v)\int_0^t H_{m_1}(X_s^{(1)})H_{m_2}(X_s^{(2)})ds}{A_t(u,v)^{1/2}}\right)$$
$$\longrightarrow 0$$

as $t \to \infty$.

We prove Theorem 1 first using Lemma 2 and leave the proof of Lemma 2 to the section 3. By Lemma 2, $Var(Y_t(u,v)) \to 1$ as $t \to \infty$. Referring to the expression (7), we may conclude that $Y_t(u,v)$ and

$$\frac{\phi(u)[1-\Phi(v)]\int_0^t X_s^{(1)}ds + [1-\Phi(u)]\phi(v)\int_0^t X_s^{(2)}ds}{A_t(u,v)^{1/2}}$$

have the same limiting distribution as t tends to ∞. Therefore the proof of the theorem is completed. Now it remains to prove Lemma 2.

3. Proof

In this section we prove Lemma 2. It is equivalent to show that

$$(9)\quad \lim_{t\to\infty}\sum_{j=2}^{\infty}\sum_{|\boldsymbol{m}|=|\boldsymbol{n}|=j}\frac{B_0(\boldsymbol{m},\boldsymbol{n},u,v)}{A_t(u,v)}$$
$$\times\sum_{k=max(0,m_1-n_2)}^{min(m_1,n_1)}\frac{m_1!m_2!n_1!n_2!}{k!(n_1-k)!(m_1-k)!(n_2-m_1+k)!}$$
$$\times\int_0^t\int_0^t r^{n_2-m_1+2k}(s_1-s_2)\rho^{m_1+n_1-2k}(s_1-s_2)ds_1ds_2=0,$$

where

$$(10)\qquad u=u(t)\sim t^{\beta/2},\ v=v(t)\sim t^{\beta/2}.\ 0<\beta<\alpha.$$

Here we note that the increasing speeds of $u(t)$ and $v(t)$ are asymptotically the same. Since $\rho(s)\le r(s)$ and $\rho(s)=o(r(s))$, $\rho(s_1-s_2)\le r(s_1-s_2)$. Therefore

$$(11)\quad \int_0^t\int_0^t r^{n_2-m_1+2k}(s_1-s_2)\rho^{m_1+n_1-2k}(s_1-s_2)ds_1ds_2$$
$$\le\int_0^t\int_0^t r^{n_2-m_1+2k+m_1+n_1-2k}(s_1-s_2)ds_1ds_2$$
$$=\int_0^t\int_0^t r^j(s_1-s_2)ds_1ds_2=2\int_0^t(t-s)r^j(s)ds.$$

Also, since $[1-\Phi(u)]\phi(v)\phi(u)[1-\Phi(v)]\int_0^t(t-s)\rho(s)ds>0$, we have

$$\frac{B_0(\boldsymbol{m},\boldsymbol{n},u,v)}{A_t(u,v)}<\frac{B_0(\boldsymbol{m},\boldsymbol{n},u,v)}{2A_t'(u,v)}.$$

By (11) the term in (9) is bounded by

$$\sum_{j=2}^{\infty}\sum_{|\boldsymbol{m}|=|\boldsymbol{n}|=j}\frac{B_0(\boldsymbol{m},\boldsymbol{n},u,v)}{A_t'(u,v)}$$
$$\times\sum_{k=max(0,m_1-n_2)}^{min(m_1,n_1)}\frac{m_1!m_2!n_1!n_2!}{k!(n_1-k)!(m_1-k)!(n_2-m_1+k)!}\int_0^t(t-s)r^j(s)ds,$$

which can be expressed as an asymptotically equivalent term, when we replace $A'_t(u,v)$ by (8), as like

(12)
$$\sum_{j=2}^{\infty}\sum_{|\boldsymbol{m}|=|\boldsymbol{n}|=j}\frac{B'_0(\boldsymbol{m},\boldsymbol{n},u,v)t^{-(j-1)\alpha}}{[1-\Phi(u)]^2\phi^2(v)+\phi^2(u)[1-\Phi(v)]^2}\cdot\frac{(m_1+m_2)!}{m_1!m_2!n_1!n_2!},$$

where $B'_0(\boldsymbol{m},\boldsymbol{n},u,v) = m_1!m_2!n_1!n_2!B_0(\boldsymbol{m},\boldsymbol{n},u,v)$ and (12) can be obtained from the following (13) and (14).

$$\text{(13)}\qquad \frac{\int_0^t(t-s)r^j(s)ds}{\int_0^t(t-s)r(s)ds}\sim t^{-(j-1)\alpha}$$

and

(14)
$$\sum_{k=max(0,m_1-n_2)}^{min(m_1,n_1)}\frac{1}{k!(n_1-k)!(m_1-k)!(n_2-m_1+k)!}=\frac{(m_1+m_2)!}{m_1!m_2!n_1!n_2!}.$$

It is not hard to show the following

$$\frac{j!}{m_1!m_2!n_1!n_2!}=\frac{(m_1+m_2)!}{m_1!m_2!n_1!n_2!}<\frac{4}{[j/2]!}\quad\text{when } j=|\boldsymbol{m}|=|\boldsymbol{n}|,$$

where $[p]$ is the interger part of p. Thus (12) is bounded by

$$\text{(15)}\qquad 4\sum_{j=2}^{\infty}\frac{1}{[j/2]!}\sum_{|\boldsymbol{m}|=|\boldsymbol{n}|=j}\frac{B'_0(\boldsymbol{m},\boldsymbol{n},u,v)t^{-(j-1)\alpha}}{[1-\Phi(u)]^2\phi^2(v)+\phi^2(u)[1-\Phi(v)]^2}.$$

LEMMA 3. *For given $\epsilon>0$, there exists a real number $T>0$ such that*

$$\frac{B'_0(\boldsymbol{m},\boldsymbol{n},u,v)t^{-(j-1)\alpha}}{[1-\Phi(u)]^2\phi^2(v)+\phi^2(u)[1-\Phi(v)]^2}<\epsilon$$

for $t>T$ and for all $j\geqslant 2$.

Proof. First note that we still assume the condition (10). To prove Lemma 3 we consider three cases for B'_0. Let $B'(k,w)=k!B(k,w)$.

CASE 1. $B_0'(\boldsymbol{m},\boldsymbol{n},u,v) = B'(0,u)B'(j,v)B'(0,u)B'(j,v)$. Then

$$
\begin{aligned}
&\frac{B_0'(\boldsymbol{m},\boldsymbol{n},u,v)t^{-(j-1)\alpha}}{[1-\Phi(u)]^2\phi^2(v)+\phi^2(u)[1-\Phi(v)]^2} \\
&= \frac{[1-\Phi(u)]H_{j-1}(v)\phi(v)[1-\Phi(u)]H_{j-1}(v)\phi(v)t^{-(j-1)\alpha}}{[1-\Phi(u)]^2\phi^2(v)+\phi^2(u)[1-\Phi(v)]^2} \\
&= \frac{[1-\Phi(u)]^2H_{j-1}^2(v)\phi^2(v)t^{-(j-1)\alpha}}{[1-\Phi(u)]^2\phi^2(v)+\phi^2(u)[1-\Phi(v)]^2} \\
&\sim \frac{[1-\Phi(t^{\beta/2})]^2\phi^2(t^{\beta/2})H_{j-1}^2(t^{\beta/2})t^{-(j-1)\alpha}}{2[1-\Phi(t^{\beta/2})]^2\phi^2(t^{\beta/2})} \\
&= H_{j-1}^2(t^{\beta/2})t^{-(j-1)\alpha}/2 \\
&= \frac{1}{2}\{t^{\beta(j-1)}t^{-(j-1)\alpha}+o(t^{(\beta-\alpha)(j-1)})\} \\
&= \frac{1}{2}\{t^{(\beta-\alpha)(j-1)}+o(t^{(\beta-\alpha)(j-1)})\}
\end{aligned}
\tag{16}
$$

the last term in (16) converges to 0 as t tends to ∞.

CASE 2. $B_0'(\boldsymbol{m},\boldsymbol{n},u,v) = B'(0,u)B'(j,v)B'(n_1,u)B'(n_2,v)$,

$$
\begin{aligned}
&\frac{B_0'(\boldsymbol{m},\boldsymbol{n},u,v)t^{-(j-1)\alpha}}{[1-\Phi(u)]^2\phi^2(v)+\phi^2(u)[1-\Phi(v)]^2} \\
&\sim \frac{[1-\Phi(t^{\beta/2})]H_{j-1}(t^{\beta/2})\phi(t^{\beta/2})H_{n_1-1}(t^{\beta/2})H_{n_2-1}(t^{\beta/2})\phi(t^{\beta/2})t^{-(j-1)\alpha}}{2[1-\Phi(t^{\beta/2})]^2\phi^2(t^{\beta/2})} \\
&= \frac{\phi(t^{\beta/2})H_{j-1}(t^{\beta/2})H_{n_1-1}(t^{\beta/2})H_{n_2-1}(t^{\beta/2})t^{-(j-1)\alpha}}{2[1-\Phi(t^{\beta/2})]} \\
&= \frac{(1/\sqrt{2\pi})e^{-t^\beta/2}(t^{\beta/2})^{2j-3}t^{-(j-1)\alpha}}{2\int_{t^{\beta/2}}^\infty \phi(s)ds} \\
&= \frac{1}{2\sqrt{2\pi}}\cdot\frac{t^{(\beta-\alpha)(j-1)}}{\int_{t^{\beta/2}}^\infty \phi(s)ds/t^{-\beta/2}e^{-t^\beta/2}}
\end{aligned}
\tag{17}
$$

Simple computation reveals that the limit of the denominator of the last term in (17) is a finite number. Therefore the last term in (17) converges to 0 as t tends to ∞.

CASE 3. $B'_0(\boldsymbol{m},\boldsymbol{n},u,v) = B'(m_1,u)B'(m_2,v)\cdot B'(n_1,u)B'(n_2,v)$,

$$(18)\qquad \begin{aligned}&\frac{B'_0(\boldsymbol{m},\boldsymbol{n},u,v)t^{-(j-1)\alpha}}{[1-\Phi(u)]^2\phi^2(v)+\phi^2(u)[1-\Phi(v)]^2}\\ &=\frac{\phi^2(u)\phi^2(v)H_{m_1-1}(u)H_{m_2-1}(v)H_{n_1-1}(u)H_{n_2-1}(v)t^{-(j-1)\alpha}}{[1-\Phi(u)]^2\phi^2(v)+\phi^2(u)[1-\Phi(v)]^2}\end{aligned}$$

By assumption we may express (18) asymptotically using $\phi^2(u) = (1/2\pi)e^{-u^2} \sim (1/2\pi)e^{-t^\beta}, \phi^2(v) = (1/2\pi)e^{-v^2} \sim (1/2\pi)e^{-t^\beta}$,

$$1-\Phi(u) = \int_u^\infty \phi(s)ds \sim \int_{t^{\beta/2}}^\infty \phi(s)ds,$$

and

$$\begin{aligned}&H_{m_1-1}(u)H_{m_2-1}(v)H_{n_1-1}(u)H_{n_2-1}(v)\\ &= u^{m_1+n_1-2}v^{m_2+n_2-2}+o(u^{m_1+n_1-2}v^{m_2+n_2-2})\\ &\sim t^{(\beta/2)(m_1+n_1-2)}t^{(\beta/2)(m_2+n_2-2)}+o(u^{m_1+n_1-2}v^{m_2+n_2-2})\\ &= t^{(\beta/2)(2j-4)}+o(u^{m_1+n_1-2}v^{m_2+n_2-2})\\ &= t^{\beta(j-2)}+o(u^{m_1+n_1-2}v^{m_2+n_2-2}).\end{aligned}$$

The first term in (18) resulted in by the substitution is

$$\frac{1}{4\pi}\cdot\frac{e^{-t^\beta}t^{-\beta}t^{(\beta-\alpha)(j-1)}}{[\int_{t^{\beta/2}}^\infty \phi(s)ds]^2}.$$

The following is not hard to show. Indeed, we may use the L'Hospital's Rule to find the finite limit.

$$(19)\qquad \lim_{t\to\infty}\frac{[\int_{t^{\beta/2}}^\infty \phi(s)ds]^2}{e^{-t^\beta}t^{-\beta}} = \frac{1}{2\pi},\quad \lim_{t\to\infty} t^{(\beta-\alpha)(j-1)} = 0.$$

Therefore

$$\lim_{t\to\infty}\frac{1}{4\pi}\cdot\frac{e^{-t^\beta}t^{-\beta}t^{(\beta-\alpha)(j-1)}}{[\int_{t^{\beta/2}}^\infty \phi(s)ds]^2} = 0.$$

Since the left side in (18) converges to 0, so does the rest of the terms.

The other cases are the same to one of the three cases above. Therefore we have completed the proof of Lemma 3.

Note that Lemma 3 holds uniformly on the integers $j \geqslant 2$. Therefore (15) is bounded by

$$4\epsilon \sum_{j=2}^{\infty} \frac{(j+1)^2}{[j/2]!}, \tag{20}$$

when t is sufficiently large. But the series in (20) converges to a finite number. This completes the proof of Lemma 2.

References

1. S. M. Berman, *High level sojourns for strongly dependent Gaussian processes.*, Z. Wahrscheinlichkeitstheorie verw. Gebiete. **50** (1979), 223-236.
2. S. M. Berman, *Sojourns of vector Gaussian processes inside and outside spheres.*, Z. Wahr scheinlichkeitstheorie verw. Gebiete. **66** (1984), 529-542.
3. P. Breuer, and P. Major, *Central limit theorems for non-linear functionals of Gaussian fields.*, J. Mult. Anal. **13** (1983), 425-441.
4. L. Giraitis, and D. Surgailis, *CLT and other limit theorems for functionals of Gaussian processes.*, Z. Wahrscheinlichkeitstheorie verw. Gebiete. **70** (1985), 191-212.
5. H. C. Ho, and T. C. Sun, *Limiting distributions of non-linear vector functions of stationary Gaussian process.*, Ann. Prob. **18, No 3** (1990), 1159-1173.
6. M. Maejima, *Some sojourn time problems for strongly dependent Gaussian processes.*, Z. Wahrscheinlichkeitstheorie verw. Gebiete **57** (1981), 1-14.
7. M. Maejima, *Some limit theorems for sojourn times of strongly dependent Gaussian processes.*, Z. Wahrscheinlichkeitstheorie verw. Gebiete **60** (1982), 359-380.
8. M. maejima, *Some sojourn time problems for 2-dimensional Gaussian processes.*, J. Mult. Anal. **18** (1986), 52-69.
9. P. Major, *Limit theorems for non-linear functionals of Gaussian sequences.*, Z. Wahrscheinlichkeitstheorie verw. Gebiete **57** (1981), 129-158.
10. T. C. Sun, *Some further results of central limit theorems for non-linear functions of normal stationary process.*, J. Math. Mech. **14** (1965), 71-85.

Department of Mathematics
Taejon University 96-3
YongWoon-dong Dong-gu
Taejon 300-716, Korea