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STRONG MAXIMAL MEANS WITH RESPECT TO NON–PRODUCT MEASURES

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§1. Introduction and description of main results

In the present article we consider multiparameter maximal averages and discover the crucial roles played by the number of parameters in their boundedness properties. The problem we shall deal with is initiated by Rubio de Francia [8] and will be in the spirit of an inductive extension to multiparameter cases, in which tools of our study rely on the theory of Harmonic Analysis on product spaces. Suppose that $d\mu$ is a complex Borel measure supported on a compact subset S of $\mathbf{R}^N$ having total mass one, $\int_S d\mu = 1$. We split variables of $\mathbf{R}^N$ arbitrarily into k-factors

$$x = (x_1, \cdots, x_k), \qquad x_j \in \mathbf{R}^{n_j}, \qquad N = n_1 + \cdots + n_k.$$

For a continuous complex-valued function f on $\mathbf{R}^N$, we consider the k-parameter family of averaging operators $\{A_{t_1,\cdots,t_k} f\}_{t_1,\cdots,t_k>0}$ defined as

$$\begin{aligned}(A_{t_1,\cdots,t_k} f)(x) &= \int_S f(x_1 - t_1\omega_1, \cdots, x_k - t_k\omega_k)\, d\mu_\omega \\ &= (f * d\mu_{t_1,\cdots,t_k})(x),\end{aligned} \tag{1-1}$$

where $d\mu_{t_1,\cdots,t_k}$ denotes the multiparameter-dilated measure

$$\int g(x)\, d\mu_{t_1,\cdots,t_k}(x) = \int g(t_1x_1, \cdots, t_kx_k)\, d\mu(x)$$

whenever g is a continuous function with compact support. For a general function, particularly, for $f \in L^p(\mathbf{R}^N)$, we are ultimately concerned about the limiting behaviors of averages with our focus on the role played by the number of parameters in this connection. To state precisely, we would like to ask the following problem in light of the Jessen-Marcinkiewicz-Zygmund strong maximal theorem.

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QUESTION. *For which p and under what conditions, does the strong differentiability with respect to* $d\mu$,

$$(A_{t_1,\cdots,t_k} f)(x) \to f(x) \quad \text{as} \quad t_1,\cdots,t_k \to 0 \quad \textit{independently}$$

hold almost everywhere and how does the number of parameters affect the results?

Equivalently, we are led directly to the continuity problem for the associated strong maximal averages

$$\mathcal{A}^* f(x) = \sup_{t_1,\cdots,t_k>0} |(A_{t_1,\cdots,t_k} f)(x)| \tag{1-2}$$

with respect to the norms of $L^p(\mathbf{R}^N)$. From the classical Fourier multiplier point of view, we observe that the strong averages are obtained by a multiparameter group of dilations for the fixed function $(d\mu)\widehat{}$ at least for Schwartz functions, that is, we may rewrite

(1-3)

$$(A_{t_1,\cdots,t_k} f)\widehat{}(\xi_1,\cdots,\xi_k) = \hat{f}(\xi_1,\cdots,\xi_k)(d\mu)\widehat{}(t_1\xi_1,\cdots,t_k\xi_k).$$

In the trivial cases when $d\mu = d\mu_1 \times \cdots \times d\mu_k$ where each Borel measure is supported in a compact subset S_j of $\mathbf{R}^{n_j}$, we have the *iterated* strong maximal means

(1-4)

$$\mathcal{N}^* f(x) = \sup_{t_1,\cdots,t_k>0} \left| \int_{S_1\times\cdots\times S_k} f(x_1 - t_1\omega_1,\cdots,x_k - t_k\omega_k)\, d\mu_1(\omega_1)\cdots d\mu_k(\omega_k) \right|.$$

In this particular instance, the positivity of averaging operators allows us to apply the one-parameter result in order to obtain the boundedness of the iterated strong maximal operators.

However, when the prescribed measure is not a product measure, the method of iteration is not available at hand, which we shall concentrate on in this article. Specifically, we shall establish the following theorem :

THEOREM 1. *Under the assumption that*

(1-5)
$$|(d\mu)^\frown(\xi_1,\cdots,\xi_k)| \le C \prod_{1\le j\le k}(1+|\xi_j|)^{-a_j}, \qquad \textit{for some} \qquad a_j > \frac{1}{2},$$

we have the a priori ineualities

(1)

$$\|\mathcal{A}^* f\|_{L^p} \le A_p \|f\|_{L^p}, \qquad \textit{if} \qquad p > p_{a,k} = 1 + \frac{k}{k-1+2(\min_j a_j)},$$

(2)

$$\|\mathcal{N}^* f\|_{L^p} \le C_p \|f\|_{L^p}, \qquad \textit{if} \qquad p > 1 + \left(2 \min_j a_j\right)^{-1}.$$

In particular, whenever $f \in L^p(\mathbf{R}^N)$, with the ranges of p prescribed as above, we have the strong differentiability with respect to $d\mu$, in the sense of the question.

Here the boundeness of $\mathcal{N}^*$ follows easily by making iterative use of the known one parameter result due to Rubio de Francia [8] and by the positivity of averaging operators. As expected, the larger the number of parameters is, the smaller the range of p is attainable since as a function of k, $p_{a,k}$ is nondecreasing. Observe that $1 < p_{a,k} < 2$ and when $k = 1$, $a_1 = a$ we are reduced to the theorem of Rubio de Francia [8].

More concretly, we obtain the following interesting result explaining the significant effect of the number of parameters involved.

COROLLARY 1. *Assume that for some $a > \frac{k}{2}$,*

$$|(d\mu)^\frown(\xi)| \le C(1+|\xi|)^{-a}, \quad \xi \in \mathbf{R}^N.$$

Then we have

$$\|\mathcal{A}^* f\|_{L^p} \le A_p \|f\|_{L^p}, \qquad \textit{if} \qquad p > 1 + \frac{k^2}{k^2-k+2a}.$$

The proof of the Corollary 1 is immediate upon estimating the decay assumption by the familiar minimal arithmetic-geometric inequality. We shall give numerous illustrating examples in the following section including the interesting maximal ellipsoidal means which generalizes the well-known maximal spherical means.

§2. Maximal means over ellipsoids and hypersurfaces

As an application of the above results we shall start with the maximal ellipsoidal means considered by L. Asgeirsson and F. John [2], [4] in their study of mean value theorems for solutions to the ultrahyperbolic equations. Formally, we start with the unit sphere S^{N-1} and dilate it by a group of k-parameters to get a family of ellipsoids with k semi-axes and we take averages of a function over ellipsoids with respect to the associated dilation invariant surface area measure (instead of the natural area element), so called ellipsoidal means. To put in detail, consider a family of ellipsoids in $\mathbf{R}^N$ described by a matrix G and a scale factor λ,

$$F(y) = y \cdot \left(\frac{G+G^t}{2}\right)^{-1} y = \lambda^2,$$

where we require the matrix $G+G^t$ to be symmetric positive definite. There exists a non-singular matrix T such that

$$T\left(\frac{G+G^t}{2}\right)T^t = I$$

and the linear substitution $z = Ty$ will then transform the ellipsoids into spheres $|z|^2 = \lambda^2$. If dS_y stands for the surface area element of an ellipsoid, we denote by $d\sigma$ the solid angle from the origin in z-space formed by the points z corresponding to the points y of dS_y under the substitution. It turns out by an elementary computation that

$$d\sigma = 2\lambda^{2-n}\left(\det\left(\frac{G+G^t}{2}\right)\right)^{-1/2}\frac{dS_y}{|\nabla F|}. \tag{2-1}$$

We now fix the scale factor λ and form the ellipsoidal means for a function f defined on $\mathbf{R}^N$ by

$$A_G f(x) = \frac{1}{\omega_N} \int_{F(y)=\lambda^2} f(x-y)\, d\sigma, \qquad \omega_N = \frac{2\pi^{N/2}}{\Gamma(N/2)},$$

and allow the matrix G as a parameter group to formulate the associated maximal ellipsoidal means

$$\text{(2-2)} \quad \mathcal{M}^* f(x) = \sup_G |A_G f(x)| = \sup_G \left| \frac{1}{\omega_N} \int_{F(y)=\lambda^2} f(x-y)\, d\sigma \right|,$$

where the supremum is taken over all elements of $G = (g_{ij})$ independently, which may be regarded as a version of the spherical means investigated by J. Bourgain, E. Stein and S. Wainger [9] in that when G is a parameter times the identity matrix it reduces to the maximal spherical means. If we take the Fourier transforms of ellipsoidal means by using the invariance of $d\sigma$ under general affine transformations, we obtain

$$\begin{aligned} (A_G f)\hat{}(\xi) &= \frac{1}{\omega_N} \int_{|z|=\lambda} e^{i\xi \cdot T^{-1} z}\, d\sigma\, \hat{f}(\xi) \\ &= 2^{(N-2)/2}\, \Gamma(N/2)\, |T^{-1}\xi|^{-(N-2)/2} J_{(N-2)/2}(|T^{-1}\xi|)\, \hat{f}(\xi) \\ &= (d\sigma)\hat{}(\lambda |T^{-1}\xi|)\, \hat{f}(\xi), \end{aligned}$$

where we observe $|T^{-1}\xi| = \sqrt{\xi \cdot G\xi}$. By rotations of axes and translations and by the invariance of $d\sigma$, we may assume that $\lambda = 1$ and G is a k-block diagonal matrix so that ellipsoids take the form

$$\sum_{j=1}^{k} \left(\frac{|y_j|}{t_j} \right)^2 = 1, \qquad y_j \in \mathbf{R}^{n_j}, \qquad j = 1, \cdots, k.$$

Applying Corollary 1 and the known decay estimates of Bessel functions, we get

THEOREM 2. *For the maximal ellipsoidal means (2-2), we have*

$$\|\mathcal{M}^*f\|_{L^p} \leq B_p\|f\|_{L^p}, \qquad p > 1 + \frac{k^2}{k^2 - k - 1 + N}, \qquad k \leq N - 2.$$

As a consequence the sequence of ellipsoidal means $\{A_G f(x)\}$ *of a function* f *converges pointwise to* $f(x)$ *almost everywhere when the ellipsoids centered at* x *shrink arbitrarily to* x *and* $f \in L^p(\mathbf{R}^N)$.

Note that when $k = 1$ we get Stein's theorem of maximal spherical means with $p > \dfrac{N}{N-1}$. Turning to other averages, we follow W. Littman [6] to consider a smooth hypersurface $S \subset \mathbf{R}^N$ with a smooth measure $d\mu$ compactly supported away from the boundary. Suppose that, for all points of $\mathrm{supp}(d\mu) \cap S$, at least α of the principal curvatures are nonzero. Let us put

$$\text{(2-4)} \qquad \mathcal{H}^*f(x) = \sup_{s_1, \cdots, s_k > 0} \left| \int_S f(x_1 - s_1\zeta_1, \cdots, x_k - s_k\zeta_k)\, d\mu_\zeta \right|.$$

It follows from the estimate of Littman, $|(d\mu)\hat{\ }(\xi)| \leq C(1 + |\xi|)^{-\alpha/2}$, that

THEOREM 3.

$$\|\mathcal{H}^*f\|_{L^p} \leq H_p\|f\|_{L^p}, \qquad p > 1 + \frac{k^2}{k^2 - k + \alpha}, \qquad k < \alpha.$$

In particular, if the hypersurface S has the nonzero Gaussian curvature everywhere, i.e., $\alpha = N - 1$ then we have the same range of p's as in the case of maximal ellipsoidal means. Indeed the relation (2-1) could be expressed entirely in terms of the Gaussian curvature of ellipsoids and the determinant of the defining matrix so that as far as the Fourier transforms for the two measures are concerned they behave according to the same decay rate.

§3. Decomposition and L^2 estimates

To the end of proving Theorem 1, we assume $k = 2$ and in this special case we write $\mathcal{A}^* = \mathcal{T}^*$, $t_1 = s, t_2 = t, a_1 = a, a_2 = b$, and $(d\mu)\widehat{\ } = m$ for notational convenience. We begin with L^2 estimates. Following [1] we decompose our operator $\mathcal{T}^*$ as follows. We fix a radial Schwartz function ϕ on $\mathbf{R}$ so that

$$\operatorname{supp}(\phi) \subset \left\{\frac{1}{2} < |t| < 2\right\}, \qquad 0 \le \phi(t) \le 1 \qquad \text{for all} \quad t,$$

and

$$\sum_{k\in\mathbf{Z}} \phi(2^{-k}t) = 1 \qquad \text{for} \quad t \neq 0.$$

Pick another auxiliary radial Schwartz function p on $\mathbf{R}$ such that

$$\operatorname{supp}(p) \subset \left\{|t| < \frac{1}{2}\right\}, \qquad p(t) = 1 \qquad \text{if} \quad |t| \le \frac{1}{3},$$

and consider the smooth Taylor polynomials associated with m

$$\begin{aligned}
\Phi_1(\xi,\eta) &= (1 - p(\xi))\, p(\eta) \sum_{|\rho|\le k_2} \left(\partial_\eta^\rho m\right)(\xi,0)\frac{\eta^\rho}{\rho!},\\
\Phi_2(\xi,\eta) &= (1 - p(\eta))\, p(\xi) \sum_{|\sigma|\le k_1} \left(\partial_\xi^\sigma m\right)(0,\eta)\frac{\xi^\sigma}{\sigma!},\\
\Phi_3(\xi,\eta) &= p(\xi)p(\eta)\Bigg\{ \sum_{|\rho|\le k_2} \left(\partial_\eta^\rho m\right)(\xi,0)\frac{\eta^\rho}{\rho!} + \sum_{|\sigma|\le k_1} \left(\partial_\xi^\sigma m\right)(0,\eta)\frac{\xi^\sigma}{\sigma!}\\
&\qquad - \sum_{|\sigma|\le k_1, |\rho|\le k_2} \left(\partial_\eta^\rho\partial_\xi^\sigma m\right)(0,0)\frac{\xi^\sigma\eta^\rho}{\sigma!\rho!}\Bigg\},\\
\Phi(\xi,\eta) &= \Phi_1(\xi,\eta) + \Phi_2(\xi,\eta) + \Phi_3(\xi,\eta),
\end{aligned}$$

where $p(\xi) = p(|\xi|)$, $p(\eta) = p(|\eta|)$. Setting $\nu(\xi,\eta) = m(\xi,\eta) - \Phi(\xi,\eta)$, we have

$$\nu(\xi,\eta) = \sum_{i,j\in\mathbf{Z}} \phi(2^{-i}\xi)\phi(2^{-j}\eta)\nu(\xi,\eta) = \sum_{i,j\in\mathbf{Z}} \nu_{ij}(\xi,\eta).$$

Note that

$$\operatorname{supp}(\nu_{ij}) \subset \left\{ (\xi,\eta) \,\middle|\, 2^{i-1} < |\xi| < 2^{i+1},\ 2^{j-1} < |\eta| < 2^{j+1} \right\}.$$

We define the Fourier multiplier operators $\{T_{ij}^{s,t}\}$ on these torus-shell regions by

$$\left(T_{ij}^{s,t}\right)\widehat{\ }(\xi,\eta) = \nu_{ij}(s\xi,t\eta)\hat{f}(\xi,\eta) \qquad \text{for} \quad s,t>0$$

and let T_{ij}^* denote its maximal operator

$$T_{ij}^* f(x,y) = \sup_{s,t>0} \left|T_{ij}^{s,t} f(x,y)\right|.$$

With the kernel $K_{s,t}(x,y) = s^{-n_1}t^{-n_2}K(x/s, y/t)$, $\hat{K} = \Phi$, we have

$$T^* f(x,y) \le \sum_{i,j\in\mathbf{Z}} T_{ij}^* f(x,y) + \sup_{s,t>0} \left|\left(K_{s,t} * f\right)(x,y)\right|.$$

Based on the standard results of one-parameter case and the easy majorizations by the sectional Hardrdy-Littlewood maximal operators as well as the strong maximal functions we can see that

$$\left\| \sup_{s,t>0} \left|\left(K_{s,t} * f\right)(x,y)\right| \right\|_{L^p(\mathbf{R}^{n_1}\times\mathbf{R}^{n_2})} \le A_p \|f\|_{L^p(\mathbf{R}^{n_1}\times\mathbf{R}^{n_2})},$$

provided

$$p > \max\left(\frac{2a+1}{2a}, \frac{2b+1}{2b}\right).$$

Now for the L^2 estimates we exploit our hypertheses by the use of square functions stemming from the following elementary facts

$$\begin{aligned}
&\left(T_{ij}^* f\right)^2 \\
\le& 2\int_0^\infty\int_0^\infty |T_{ij}^{s,t} f|\,|\partial_s\partial_t T_{ij}^{s,t} f|\, ds dt + 2\int_0^\infty\int_0^\infty |\partial_s T_{ij}^{s,t} f|\,|\partial_t T_{ij}^{s,t} f|\, ds dt \\
\le& 2\left(\int_0^\infty\int_0^\infty |T_{ij}^{s,t} f(x,y)|^2 \frac{dsdt}{st}\right)^{1/2} \left(\int_0^\infty\int_0^\infty |st\partial_s\partial_t T_{ij}^{s,t} f(x,y)|^2 \frac{dsdt}{st}\right)^{1/2} \\
&+ 2\left(\int_0^\infty\int_0^\infty |s\partial_s T_{ij}^{s,t} f(x,y)|^2 \frac{dsdt}{st}\right)^{1/2} \left(\int_0^\infty\int_0^\infty |t\partial_t T_{ij}^{s,t} f(x,y)|^2 \frac{dsdt}{st}\right)^{1/2} \\
=& 2\left\{ \left(G_{ij} f(x)\right)\left(\tilde{G}_{ij} f(x)\right) + \left(S_{ij} f(x)\right)\left(\tilde{S}_{ij} f(x)\right) \right\}
\end{aligned}$$

which will give by the Plancherel Theorem that

LEMMA 1. *For integers i, j,*

$$\|T_{ij}^* f\|_{L^2(\mathbf{R}^{n_1}\times\mathbf{R}^{n_2})} \le C\Delta_{ij}\|f\|_{L^2(\mathbf{R}^{n_1}\times\mathbf{R}^{n_2})}$$

where

$$\Delta_{ij} = \begin{cases} 2^{i(\frac{1}{2}-a)+j(\frac{1}{2}-b)}, & \text{for } i,j \ge 0 \\ 2^{i(\frac{1}{2}-a)+j}, & \text{for } i \ge 0, j < 0 \\ 2^{i+j(\frac{1}{2}-b)}, & \text{for } i < 0, j \ge 0 \\ 2^{i+j} & \text{for } i,j < 0. \end{cases}$$

Proof. See [1] for the details.

§4. The Journé class of Calderón-Zygmund operators on product domains

Let T be an $L^2(\mathbf{R}^N)$ bounded linear integral operator defined by $Tf(x) = \int K(x,y)f(y)\,dy$. If the kernel K satisfies that

$$\text{(4-1)} \qquad \int_{|x-y|>\rho|y-z|} |K(x,y) - K(x,z)|dz \le C\rho^{-\delta}$$

for all $\rho \ge 2$ and for some $\delta > 0$, then T is called a Calderón-Zygmund operator and the Calderón-Zygmund norm of T is defined to be

$$|T|_{CZ} = \|T\|_{L^2,L^2} + \inf\{\, C \;:\; (4\text{-}1) \quad \text{holds}\}.$$

In [5], Journé extends Calderón-Zygmund operators to product domains $\mathbf{R}^{n_1} \times \cdots \times \mathbf{R}^{n_k}$ by defining the so-called Journé class and proves such operators map L^∞ boundedly to $BMO(\mathbf{R}^{n_1} \times \cdots \times \mathbf{R}^{n_k})$. Following [4] we describe his class of operators in the case when $k = 2$, $n_1 + n_2 = N$. For an integral operator T given by

$$Tf(x_1,x_2) = \iint K(x_1,y_1,x_2,y_2)f(y_1,y_2)\,dy_1\,dy_2,$$

we fix x_1, y_1 (respectively x_2, y_2) in the kernel $K(x_1,y_1,x_2,y_2)$ to acquire the kernel $K_1(x_1,y_1)$ ($K_2(x_2,,y_2)$) and let $\tilde{T}(x_1,y_1)$ (respectively. $\tilde{T}(x_2,y_2)$) be the integral operator defined by this kernel. T is said to be in the Journé class if it is bounded in L^2 and verifies

$$\text{(4-2)} \qquad \int_{|x_j-y_j|>\gamma|y_j-y_j'|} |\tilde{T}(x_j,y_j) - \tilde{T}(x_j,y_j')|_{CZ}\,dx_j \le C_j\gamma^{-\delta}, \qquad j = 1,2$$

for all $\gamma \geq 2$ and for some $\delta > 0$. As before we may define the Calderón-Zygmund norm by

$$|T|_{CZ} = \|T\|_{2,2} + \sum_{j=1}^{2} \inf\{ C_j : \text{(4-2) holds}\}.$$

For a general product domains of an arbitrary number of factors $\mathbf{R}^{n_1} \times \cdots \times \mathbf{R}^{n_k}$, $k \geq 3$, the Journé class of operators and the associated CZ-norms are defined inductively. In consideration of the actions of these operators on Hardy spaces of product domains, R. Fefferman [3] shows that for a product domain of two factors they map $H^1(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ continuosly into $L^1(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ by making use of his celebrated boundedness criterion. Later in [7], J. Pipher proves the same continuity result in the case of an arbitrary product domains consisting of more than two factors by extending Journé's geometric covering lemma to higher dimensions (see [3], [7]). Here the Hardy space $H^1(\mathbf{R}^{n_1} \times \cdots \times \mathbf{R}^{n_k})$ is best characterized by the atomic decomposition of A. Channg and R. Fefferman. By an atom we mean a function a_Ω supported on an open set Ω of finite measure such that $a_\Omega = \sum_{S \in \mathcal{M}(\Omega)} \alpha_S$, where $\mathcal{M}(\Omega)$ denotes the maximal class of dyadic rectangles (product of cubes) $S \subset \Omega$. The rectangle atoms α_S are supported in a 2-fold enlargement of S and their integrals vanish when evaluated in each variables separately. Moreover,

$$\|a_\Omega\|_{L^2}^2 \leq |\Omega|^{-1} \qquad \text{and} \qquad \sum_{S \in \mathcal{M}(\Omega)} \|\alpha_S\|_{L^2}^2 \leq |\Omega|^{-1}.$$

The product Hardy space H^1 can be described as the subspace of L^1 consisting of all functions

$$f = \sum_{\Omega} \lambda_\Omega \, a_\Omega, \qquad \|f\|_{H^1} = \inf \sum_{\Omega} |\lambda_\Omega| < \infty.$$

Since the continuity constants will play a significant role in deciding the range of p's in our interpolation argument later. we shall state carefully what they proved in the following form :

PROPOSITION. *If T belongs to the Journé class, then we have*

$$\|Tf\|_{L^1(\mathbf{R}^{n_1}\times\cdots\times\mathbf{R}^{n_k})} \le C|T|_{CZ(\mathbf{R}^{n_1}\times\cdots\times\mathbf{R}^{n_k})}\|f\|_{H^1(\mathbf{R}^{n_1}\times\cdots\times\mathbf{R}^{n_k})}.$$

Let us now focus on the maximal operators considered in the section §3, with $\hat{K}_{ij} = \nu_{ij}$,

$$T_{ij}^* f(x,y) = \sup_{s,t>0} \left|\left(K_{ij}^{s,t} * f\right)(x,y)\right|, \quad K_{ij}^{s,t}(x,y) = s^{-n_1}t^{-n_2}K_{ij}(x/s, y/t).$$

We may view these operators as linear L^2 bounded integral operators taking the complex-valued functions into the l^∞-valued functions, where the corresponding kernels $K_{ij}(x,y) = \left(K_{ij}^{s,t}(x,y)\right)_{s,t>0}$ are Bochner-Lebesgue strongly measurable functions defined on $\mathbf{R}^{n_1}\times\mathbf{R}^{n_2}$ and with values in the space of all bounded linear operators between two Banach spaces given above endowed with the operator norm $\|\cdot\|$. With this in mind we shall show that T_{ij}^* belongs to the vector-valued Journé class and hence set up the (H^1, L^1) inequality along careful computations of CZ-norms. It turns out that the negative integer cases do not affect our interpolation results and therefore we shall deal only with the case when $i, j > 0$. It follows by our decompositions that with $\hat{\varphi} = \phi$

$$K_{ij}(x,y) = \left(2^{i\,n_1 + j\,n_2}\left\{\varphi_i(x)\varphi_j(y)\right\} * d\mu\right)(x,y),$$

where $\varphi_i(x) = \varphi(2^i x)$, $\varphi_j(y) = \varphi(2^j y)$. In the first place, in accordance with the notations of (4-2), we apply Young's inequality to see that

$$\int_{|x_1-y_1|>\gamma|y_1-y_1'|} \|\tilde{T}_{ij}^*(x_1,y_1) - \tilde{T}_{ij}^*(x_1,y_1')\|_{2,2}\,dx_1$$

is, after changing variables, bounded by

$$J_{ij}(|h|) = \int_{|x|>\gamma|h|}\int \left\|K_{ij}^{s,t}(x,y) - K_{ij}^{s,t}(x-h,y)\right\|\,dy\,dx$$

$$\le \int_{|x|>\gamma|h|}\int \left\{\sup_{s,t>0}\left|K_{ij}^{s,t}(x,y) - K_{ij}^{s,t}(x-h,y)\right|\right\}\,dy\,dx. \tag{4-3}$$

We shall show that

LEMMA 2.

$$\sup_{h\in\mathbf{R}^{n_1}} \{ J_{ij}(|h|) \} \le C\,\|d\mu\|\,2^{2i}\,\gamma^{-1/2}.$$

Proof. For any fixed $t>0$ and $x\neq 0$, since $K_{ij}^{s,t}\to 0$ as $s\to 0$, we have

$$\begin{aligned} K_{ij}^{s,t}(x,y) &= \int_0^s \frac{\partial}{\partial s}K_{ij}^{s,t}(x,y)\,ds \\ &= -\int_0^s \left(n_1 K_{ij}^{s,t}(x,y)+\tilde{K}_{ij}^{s,t}(x,y)\right)\frac{ds}{s}, \end{aligned}$$

where $\tilde{K}_{ij}(x,y)=\nabla_1 K_{ij}(x,y)\cdot x$. Therefore

$$\begin{aligned} J_{ij}(|h|) &\le \int_0^\infty \int_{|x|>\gamma|h|}\int \sup_{t>0}|K_{ij}^{s,t}(x,y)-K_{ij}^{s,t}(x-h,y)|\,dy\,dx\,\frac{ds}{s} \\ &+\int_0^\infty \int_{|x|>\gamma|h|}\int \sup_{t>0}|\tilde{K}_{ij}^{s,t}(x,y)-\tilde{K}_{ij}^{s,t}(x-h,y)|\,dy\,dx\,\frac{ds}{s} \end{aligned} \tag{4-4}$$

and we handle the first integral. Observe that $\sup_{t>0}|K_{ij}^{s,t}(x,y)-K_{ij}^{s,t}(x-h,y)|$ is majorized by

$$\begin{aligned} 2^{in_1+jn_2}s^{-n_1}\int_S &\left|\varphi_i(\frac{x}{s}-w)-\varphi_i(\frac{x-h}{s}-w)\right| \\ &\left\{\sup_{t>0} t^{-n_2}\left|\varphi_j(\frac{y}{t}-z)\right|\right\} d|\mu|(w,z) \end{aligned}$$

and by the familiar square function argument related to the Schwartz function that for any $z\in S$

$$\int_{\mathbf{R}^{n_2}}\left\{\sup_{t>0} 2^{j\,n_2}t^{-n_2}\left|\varphi_j(\frac{y}{t}-z)\right|\right\} dy \le C.$$

It follows consequently that the first portion of (4-4) is dominated by a constant times

$$\int_S\int_0^\infty\int_{|x|>\frac{\gamma|h|}{s}} 2^{in_1}\left|\varphi\left(2^i(x-w)\right)-\varphi\left(2^i(x-\frac{h}{s}-w)\right)\right| dx\,\frac{ds}{s}\,d|\mu|(w,z).$$

We now break the integration with respect to $\frac{ds}{s}$ into two parts by introducing the intermediate integrating limit $\sqrt{\gamma}|h|$ and calculate each part separately. In dealing with the first integrals, we note that it is bounded by

$$\begin{aligned}
&2\int_S d\,|\mu|(w,z)\int_0^{\sqrt{\gamma}|h|}\int_{|x|>\frac{\gamma|h|}{2s}} 2^{i\,n_1}\left|\varphi\left(2^i(x-w)\right)\right|dx\,\frac{ds}{s}\\
&\le 2\int_S d\,|\mu|(w,z)\int_{|x|>\frac{\sqrt{\gamma}}{2}} 2^{i\,n_1}\left|\varphi\left(2^i(x-w)\right)\right|\log\left(\frac{2|x|}{\sqrt{\gamma}}\right)dx\\
&\le C\,\gamma^{-1/2}\int_S d\,|\mu|(w,z)\int_{|x|>2^{i-3}}\left|\varphi\left(x-2^i\,w\right)\right|\left|2^{-i}\,x\right|\,dx\le C\,\|d\mu\|\,\gamma^{-1/2}.
\end{aligned}$$

The second portion is less than or equal to

$$\begin{aligned}
&\int_S d\,|\mu|(w,z)\int_{\sqrt{\gamma}|h|}^{\infty}\int_{|x|>\frac{\gamma|h|}{s}} 2^{i\,n_1}\left|\varphi\left(2^i(x-w)\right)-\varphi\left(2^i(x-\frac{h}{s}-w)\right)\right|dx\,\frac{ds}{s}\\
&=\|d\mu\|\int_{\sqrt{\gamma}}^{\infty}\int_{|x|>\frac{\gamma}{s}} 2^{i\,n_1}\left|\varphi\left(2^i\,x\right)-\varphi\left(2^i(x-\frac{h}{s|h|})\right)\right|dx\,\frac{ds}{s}\\
&=\|d\mu\|\int_0^{\gamma^{-1/2}}\int_{|x|>\gamma\,s} 2^{i\,n_1}\left|\varphi\left(2^i\,x\right)-\varphi\left(2^i(x-s\frac{h}{|h|})\right)\right|dx\,\frac{ds}{s}\\
&\le\|d\mu\|\int_0^{\gamma^{-1/2}}\int_{\mathbf{R}^{n_1}}\left|\varphi(x)-\varphi\left(x-2^i\,s\,\frac{h}{|h|})\right)\right|dx\,\frac{ds}{s}\\
&\le C\,\|d\mu\|\int_0^{\gamma^{-1/2}}\min\left(1,2^i\,s\right)\frac{ds}{s}\le C\,\|d\mu\|\,2^i\,\gamma^{-1/2}.
\end{aligned}$$

We could estimate the latter part in (4-4) by catching that

$$\begin{aligned}
\tilde{K}_{ij}(x,y)&=\nabla_1(K_{ij})(x,y)\cdot x\\
&=2^i\sum_{|\alpha|=1}x^\alpha\left\{2^{i\,n_1+j\,n_1}(\partial_x^\alpha\varphi)_i(x)\varphi_j(y)\,*\,d\,\mu\right\}(x,y)
\end{aligned}$$

is essentially like K_{ij} except the constant factor 2^i. This proves the lemma. □

Next toward computing the CZ-norm of T^*_{ij} we have to consider estimating, after changing variables,

$$\begin{aligned}
\text{(4-5)}\quad&\int_{|y|>\gamma|k|}\sup_{s,t>0}\Big|\,K_{ij}^{s,t}(x,y)-K_{ij}^{s,t}(x-h,y)\\
&\qquad-\left\{\,K_{ij}^{s,t}(x,y-k)-K_{ij}^{s,t}(x-h,y-k)\,\right\}\Big|\,dy
\end{aligned}$$

in connection with (4-1). Note here that the integrand is bounded above by $2^{i\,n_1+j\,n_2}\,s^{-n_1}\,t^{-n_2}$ times

$$\int_S d\,|\mu|(w,z)\left|\varphi_i\left(\frac{x}{s}-w\right)-\varphi_i\left(\frac{x-h}{s}-w\right)\right| \left|\varphi_j\left(\frac{y}{t}-z\right)-\varphi_j\left(\frac{y-k}{t}-z\right)\right|$$

and implementing the same program as in the previous lemma shows (4-5) is less than or equal to $\gamma^{-1/2}$ times

$$C\,2^{2j}\,2^{i\,n_1}\sup_{s>0}\left\{s^{-n_1}\int_S d\,|\mu|(w,z)\left|\varphi_i\left(\frac{x}{s}-w\right)-\varphi_i\left(\frac{x-h}{s}-w\right)\right|\right\}.$$

Integrating this expression over $\{\,x\in\mathbf{R}^{n_1}\,:\,\gamma|x|>|h|\,\}$ by using the same scheme, we find the other integral portion of CZ-norm associated with the integral operator $\tilde{T}^*_{ij}(x_1,y_1)-\tilde{T}^*_{ij}(x_1,y_1')$ is less than $C\,2^{2i+2j}\,\|d\mu\|\,\gamma^{-1/2}$. Of course we could easily estimate the other expression appearing in (4-2). It is no restriction to assume that $\|d\mu\|\le 1$ and therefore what we have shown is that our maximal operators belong to the vector-valued Journé class with their CZ-norms do not exceed some uniform constant times 2^{2i+2j}. In summary, we state

LEMMA 3. *The maximal operators T^*_{ij} map $H^1(\mathbf{R}^{n_1}\times\mathbf{R}^{n_2})$ continuously into $L^1(\mathbf{R}^{n_1}\times\mathbf{R}^{n_2})$ and verifies that*

$$\|T^*_{ij}f\|_{L^1(\mathbf{R}^{n_1}\times\mathbf{R}^{n_2})}\le C\,2^{2i+2j}\,\|f\|_{H^1(\mathbf{R}^{n_1}\times\mathbf{R}^{n_2})},\qquad i,j\ge 0.$$

§5. Proofs of the main theorem

We now interpoate Lemma 1 and Lemma 4 to find that when $i,j\ge 0$, we obtain

$$\|T^*_{ij}\,f\|_p\le C\,2^{i\{2-\theta(a+3/2)\}+j\{2-\theta(b+3/2)\}}\,\|f\|_p,$$
$$\frac{1}{p}=1-\frac{\theta}{2},\qquad 0<\theta<1,$$

which shows the correct range of p's upon considering the necessary conditions of convergence for the relevant geometric series's. Hence the proof of the Theorem 1 is complete when $k = 2$. For general $k \geq 3$ it is clear that we could follow the same machinery inductively. To describe in detail, we set out decomposing and majorizing $\mathcal{A}^*$ by the integer-tuple $(i_1, \cdots, i_k)$ sum of maximal Fourier multipliers $T^*_{i_1,\cdots,i_k}$ defined on the dyadic building-blocks

$$\left\{ \xi = (\xi_1, \cdots, \xi_k) : 2^{i_j} < |\xi_j| < 2^{i_j+1}, \qquad j = 1, \cdots, k \right\},$$

which we could control by 2^k-set of square functions. It is now transparent that we eventually end up with estimates

$$\begin{aligned} \left\|T^*_{i_1,\cdots,i_k}\right\|_{2,2} &\leq C\, 2^{i_1(1/2-a_1)+\cdots+i_k(1/2-a_k)} \\ \left|T^*_{i_1,\cdots,i_k}\right|_{CZ} &\leq C\, 2^{k(i_1+\cdots+i_k)}, \qquad \text{when} \qquad i_1, \cdots, i_k \geq 0. \end{aligned}$$

Interpolating and summing suitable series's, we finish the proof.

REMARKS. Our method of proof reveals that the assumption (1-5) is required only to ensure the L^2 boundedness as expected. In Theorem 2 and 3, we were interested in the case of full dilations, that is, when $k = N$, but under the present setting we could not resolve this. Overall, when the decay factor $a \leq k/2$ our machinery breaks down, which results mainly from the nature of our square functions.

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