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CLASS FUNCTION TABLE MATRIX OF FINITE GROUPS

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1. Introduction

Let G be a finite group with k distinct conjugacy classes $C_1, C_2, \cdots, C_k$ and F an algebraically closed field such that $\mathrm{char}(F) \nmid |G|$. We denoted by $Irr_F(G)$ the set of all irreducible F-characters of G and $Cf_F(G)$ the set of all class functions of G into F. Then $Cf_F(G)$ is a commutative F-algebra with an F-basis $Irr_F(G) = \{\chi_1, \chi_2, \cdots, \chi_k\}$. Thus the map

$$(\ ,\ ) : Cf_F(G) \times Cf_F(G) \to F$$

defined by

$$(\theta, \eta) = \frac{1}{|G|} \sum_{x \in G} \theta(x)\, \eta(x^{-1})$$

is a nondegenerate symmetric bilinear form. For $\theta \in Cf_F(G)$, define $\overline{\theta} : G \to F$ by $\overline{\theta}(x) = \theta(x^{-1})$. Then $\overline{\theta} \in Cf_F(G)$ and $\overline{\overline{\theta}} = \theta$, $\overline{\theta + \eta} = \overline{\theta} + \overline{\eta}$, $\overline{\theta\eta} = \overline{\theta}\overline{\eta}$, $\overline{1}_G = 1_G$ where $\theta, \eta \in Cf_F(G)$ and 1_G is the principal character.

Define $T : Cf_F(G) \to End_F(Cf_F(G))$ by

$$T(\theta)(\eta) = \overline{\theta}\eta$$

for $\theta, \eta \in Cf_F(G)$. Then T is a faithful representation of F-algebra $Cf_F(G)$.

Let $M : Cf_F(G) \to M_k(F)$ be a matrix representation of $Cf_F(G)$ afforded by T relative to the ordered F-basis $Irr_F(G) = \{\chi_1, \chi_2, \cdots, \chi_k\}$.

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Since $T(\theta)(\chi_i) = \overline{\theta}\chi_i = \sum_{t=1}^{k}(\overline{\theta}\chi_i, \chi_t)\chi_t = \sum_{t=1}^{k}(\theta\chi_t, \chi_i)\chi_t$, we have

$$M(\theta) = (m_{ij}),$$

where $m_{ij} = (\theta\chi_i, \chi_j)$.

For a linear transtormation f of an F-vector space V, let $[f]_\alpha^\beta$ be a matrix of f relative to the ordered F-basis α and β of V. Then $[g]_\alpha^\beta [f]_\gamma^\alpha = [g \circ f]_\gamma^\beta$. Of course, $M(\theta) = [T(\theta)]_{Irr_F(G)}^{Irr_F(G)}$.

Define $T^* : Cf_F(G) \to End_F(Cf_F(G))$ by

$$T^*(\theta)(\eta) = \theta\eta.$$

Then T^* is a faithful representation of $Cf_F(G)$. Let M^* be a matrix representation afforded by T^* relative to the ordered F-basis $Irr_F(G)$ $= \{\chi_1, \chi_2, \cdots, \chi_k\}$. Then since $T^*(\theta)(\chi_i) = \theta\chi_i = \sum_{t=1}^{k}(\theta\chi_i, \chi_t)\chi_t$, we have

$$M^*(\theta) = [T^*(\theta)]_{Irr_F(G)}^{Irr_F(G)} = M(\theta)^t = M(\overline{\theta}) = [T(\overline{\theta})]_{Irr_F(G)}^{Irr_F(G)}.$$

Since M is a monomorphism of F-algebras, we obtain the following.

LEMMA 1.1. *For $\theta, \eta \in Cf_F(G)$,*

(1) *$M(\theta+\eta) = M(\theta)+M(\eta)$, and so $M(n\theta) = nM(\theta)$ for positive integer n*
(2) *$M(\theta\eta) = M(\theta)M(\eta)$, and so $M(\theta^n) = M(\eta)^n$ for positive integer n*
(3) *$M(a\theta) = aM(\theta)$ for $a \in F$*
(4) *$M(1_G) = 1$ where 1_G is a principal character and 1 is an dentity matrix*
(5) *$ImM = \{M(\theta) \mid \theta \in Cf_F(G)\}$ is a commutative F-algebra with an F-basis $\{M(\chi_1), M(\chi_2), \cdots, M(\chi_k)\}$.*

REMARK. In the case that F is the complex field $\mathbb{C}$, for each $\mathbb{C}$-character θ we have $\overline{\theta}(x) = \theta(x^{-1}) = \overline{\theta(x)}$. Hence $M^*(\theta) = \overline{M(\theta)^*}$.

2. The class function table matrix

Let F be an algebraically closed field with char$(F) = 0$. The class function table of the ordered F-basis $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ of $Cf_F(G)$ is given by the following form

	C_1	C_2	$\ldots$	C_k
α_1	$\alpha_1(C_1)$	$\alpha_1(C_2)$	$\ldots$	$\alpha_1(C_k)$
α_2	$\alpha_2(C_1)$	$\alpha_2(C_2)$	$\ldots$	$\alpha_2(C_k)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
α_k	$\alpha_k(C_1)$	$\alpha_k(C_2)$	$\ldots$	$\alpha_k(C_k)$.

DEFINITION. The class function table matrix X_α of G with respect to the ordered F-basis $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ of $Cf_F(G)$ is the form

$$X_\alpha = (\alpha_i(C_j))_{k\times k}\,.$$

The character table matrix $X = (\chi_i(C_j))_{k\times k}$ of G is the class function table matrix with respect to the ordered F-basis $Irr_F(G) = \{\chi_1, \chi_2, \cdots, \chi_k\}$.

Let

$$\overline{X} = (\overline{\chi}_i(C_j))_{k\times k}\,.$$

Then X and $\overline{X}$ are invertible. Define $f_i : G \to F$ by

$$f_i(x_j) = \begin{cases} 1 & (x_j \in C_i) \\ 0 & (x_j \notin C_i). \end{cases}$$

Then $\alpha = \{f_1, f_2, \cdots, f_k\}$ is an F-basis of $Cf_F(G)$ and $\theta = \sum_{i=1}^{k} \theta(C_i)f_i$ for $\theta \in Cf_F(G)$.

Define $h_i : G \to F$ by

$$h_i(x_j) = \delta_{ij}\frac{|G|}{|C_i|} \quad (x_j \in C_j)\,.$$

Then $\beta = \{h_1, h_2, \cdots, h_k\}$ is an F- basis of $Cf_F(G)$ and $h_i = \sum_{t=1}^{k} \overline{\chi}_t(C_i)\chi_t$ and $f_i = \frac{|C_i|}{|G|}h_i$.

Since $T(\theta)(\chi_i) = \overline{\theta}\chi_i = \sum_{t=1}^{k}(\overline{\theta}\chi_i, \chi_t)\chi_t = \sum_{t=1}^{k}(\theta\chi_t, \chi_i)\chi_t$, we have

$$M(\theta) = (m_{ij}),$$

where $m_{ij} = (\theta\chi_i, \chi_j)$.

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Define $T^* : Cf_F(G) \to End_F(Cf_F(G))$ by

$$T^*(\theta)(\eta) = \theta\eta.$$

Then T^* is a faithful representation of $Cf_F(G)$. Let M^* be a matrix representation afforded by T^* relative to the ordered F-basis $Irr_F(G) = \{\chi_1, \chi_2, \cdots, \chi_k\}$. Then since $T^*(\theta)(\chi_i) = \theta\chi_i = \sum_{t=1}^{k}(\theta\chi_i, \chi_t)\chi_t$, we have

$$M^*(\theta) = [T^*(\theta)]_{Irr_F(G)}^{Irr_F(G)} = M(\theta)^t = M(\overline{\theta}) = [T(\overline{\theta})]_{Irr_F(G)}^{Irr_F(G)}.$$

Since M is a monomorphism of F-algebras, we obtain the following.

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(5) *$ImM = \{M(\theta) \mid \theta \in Cf_F(G)\}$ is a commutative F-algebra with an F-basis $\{M(\chi_1), M(\chi_2), \cdots, M(\chi_k)\}$.*

REMARK. In the case that F is the complex field $\mathbb{C}$, for each $\mathbb{C}$-character θ we have $\overline{\theta}(x) = \theta(x^{-1}) = \overline{\theta(x)}$. Hence $M^*(\theta) = \overline{M(\theta)^*}$.

it follows that

$$\begin{aligned}\overline{X}^{\,-1}M(\overline{\theta})\overline{X} &= \operatorname{diag}(\theta(C_1),\theta(C_2),\cdots,\theta(C_k)) \text{ and}\\ \overline{X}^{\,-1}M(\theta)\overline{X} &= \operatorname{diag}(\overline{\theta}(C_1),\overline{\theta}(C_2),\cdots,\overline{\theta}(C_k)).\end{aligned}$$

Thus we have the following theorem.

THEOREM 2.1. *Let M be a matrix representation of $Cf_F(G)$ affored by the representation $T : Cf_F(G) \to End_F(Cf_F(G))$ defined by $T(\theta)(\eta) = \overline{\theta}\eta$ for $\theta,\eta \in Cf_F(G)$. Then*

$$X^{-1}M(\theta)X = \overline{X}^{\,-1}M(\overline{\theta})\overline{X} = diag(\theta(C_1),\theta(C_2),\cdots,\theta(C_k))$$

and

$$X^{-1}M(\overline{\theta})X = \overline{X}^{\,-1}M(\theta)\overline{X} = diag(\overline{\theta}(C_1),\overline{\theta}(C_2),\cdots,\overline{\theta}(C_k)).$$

COROLLARY 2.2. *Let M be a matrix representation of $Cf_F(G)$ affored by the representation $T : Cf_F(G) \to End_F(Cf_F(G))$ defined by $T(\theta)(\eta) = \overline{\theta}\eta$ for $\theta,\eta \in Cf_F(G)$. Let $D = \frac{1}{|G|}\operatorname{diag}(|C_1|,|C_2|,\cdots,|C_k|)$. Then $XD\overline{X}^{\,t} = I$.*

Proof. Since $\chi_j = \sum_{i=1}^{k}\chi_j(C_i)f_i$, we have $[T(1_G)]^{\alpha}_{Irr_F(G)} = X^t$. From $\{[T(1_G)]^{Irr_F(G)}_{\beta}[T(1_G)]^{\beta}_{\alpha}[T(1_G)]^{\alpha}_{Irr_F(G)}\}^t = [T(1_G)]^{Irr_F(G)}_{Irr_F(G)}$, it follows that $XD\overline{X}^{\,t} = I$.

In Theorem 2.1, we have

$$\{\theta(C_1),\theta(C_2),\cdots,\theta(C_k)\} = \{\overline{\theta}(C_1),\overline{\theta}(C_2),\cdots,\overline{\theta}(C_k)\}.$$

Hence, in Theorem 2.1, $\theta(C_1),\theta(C_2),\cdots,\theta(C_k)$ are eigenvalues of both $M(\theta)$ and $M(\overline{\theta})$, and so the polynomial $\prod_{i=1}^{m}(x-\theta(C_i))$ is the characteristic polynomial of both $M(\theta)$ and $M(\overline{\theta})$.

Let D_k be the set of all $k\times k$ diagonal matrices over F and $XD_kX^{-1} = \{X \vartriangle X^{-1} \mid \vartriangle \in D_k\}$. Then XD_kX^{-1} is an F-vector space of a dimension k. For $M(\theta) \in ImM$, we have $M(\theta) \in XD_kX^{-1}$ by Theorem 2.1. Hence $ImM \subseteq XD_kX^{-1}$. Since ImM is an F-vector space of dimension k, it follows that $ImM = XD_kX^{-1}$.

For conjugacy classes $C_1, C_2, \cdots, C_k$ of G, there is a permutation σ such that $\sigma(i) = j$ if $C_i^{-1} = C_j$. Of course, $\nu = \{h_{\sigma(1)}, h_{\sigma(2)}, \cdots, h_{\sigma(k)}\}$ is an F- basis of $Cf_F(G)$ and $h_{\sigma(i)} = \sum_{t=1}^{k} \chi_t(C_i)\chi_t$.

Therefore,

$$[T(1_G)]_\alpha^\beta = \frac{1}{|G|}\mathrm{diag}(|C_1|, |C_2|, \cdots, |C_k|).$$
$$[T(1_G)]_\nu^{Irr_F(G)} = X\,[T(1_G)]_\beta^{Irr_F(G)} = \overline{X}.$$

Of course, $[T(1_G)]_{Irr_F(G)}^\nu = X^{-1}$ and $[T(1_G)]_{Irr_F(G)}^\beta = \overline{X}^{-1}$. And for any ordered F-basis Γ of $Cf_F(G)$, we have $[T(1_G)]_\Gamma^\Gamma = I = M(1_G)$.

From

$$T(\theta)(f_i) = \overline{\theta} f_i = \overline{\theta}(C_i) f_i,\ T(\overline{\theta}) f_i = \theta(C_i) f_i,$$
$$T(\theta)(h_i) = \overline{\theta}(C_i) h_i,\ T(\overline{\theta}) h_i = \theta(C_i) h_i,$$
$$T(\theta)(h_{\sigma(i)}) = \overline{\theta}(C_{\sigma(i)}) h_{\sigma(i)} = \theta(C_i) h_{\sigma(i)} \text{ and } T(\overline{\theta})(h_i) = \theta(C_i) h_i,$$

it follows that

$$[T(\theta)]_\alpha^\alpha = [T(\theta)]_\beta^\beta = \mathrm{diag}(\overline{\theta}(C_1), \overline{\theta}(C_2), \cdots, \overline{\theta}(C_k)) \text{ and}$$
$$[T(\theta)]_\nu^\nu = \mathrm{diag}(\theta(C_1), \theta(C_2), \cdots, \theta(C_k)) = [T(\overline{\theta})]_\alpha^\alpha = [T(\theta)]_\beta^\beta.$$

Since

$$[T(1_G)]_{Irr_F(G)}^\nu [T(\theta)]_{Irr_F(G)}^{Irr_F(G)} [T(1_G)]_\nu^{Irr_F(G)} = [T(\theta)]_\nu^\nu \text{ and}$$
$$[T(1_G)]_{Irr_F(G)}^\nu [T(\overline{\theta})]_{Irr_F(G)}^{Irr_F(G)} [T(1_G)]_\nu^{Irr_F(G)} = [T(\overline{\theta})]_\nu^\nu,$$

we have

$$X^{-1} M(\theta) X = \mathrm{diag}(\theta(C_1), \theta(C_2), \cdots, \theta(C_k)) \text{ and}$$
$$X^{-1} M(\overline{\theta}) X = \mathrm{diag}(\overline{\theta}(C_1), \overline{\theta}(C_2), \cdots, \overline{\theta}(C_k)).$$

From

$$[T(1_G)]_{Irr_F(G)}^\beta [T(\overline{\theta})]_{Irr_F(G)}^{Irr_F(G)} [T(1_G)]_\beta^{Irr_F(G)} = [T(\overline{\theta})]_\beta^\beta \text{ and}$$
$$[T(1_G)]_{Irr_F(G)}^\beta [T(\theta)]_{Irr_F(G)}^{Irr_F(G)} [T(1_G)]_\beta^{Irr_F(G)} = [T(\theta)]_\beta^\beta,$$

$\cdots, + a_0$ is the minimal polinomial of $T^*(\theta)$ iff $q(x)$ is the minimal polinomial of $\theta \in Cf_F(G)$.

Let $q(x) = x^r + a_{r-1}x^{r-1} + \cdots, + a_0$ be the minimal polynomial of $\theta \in Cf_F(G)$. If $b_0 1_G + b_1\theta + \cdots, + b_{r-1}\theta^{r-1} = 0$, then $f(x) = b_0 + b_1 x + \cdots, + b_{r-1}x^{r-1}$ is a polynomial having θ as a root. Therefore, $q(x)|f(x)$. Since deg $f(x) <$ deg $q(x)$, this yields that $f(x) = 0$. That is, $b_0 = b_1 =, \cdots, = b_{r-1} = 0$. Hence $\{1_G, \theta^1, \theta^2, \cdots, \theta^{r-1}\}$ is independent. For any $b_0 1_G + b_1\theta + \cdots, + b_m\theta^m \in Cf_F(G)$, let $g(x) = b_0 + b_1 x + \cdots, + b_m x^m$. Then $g(x) = q(x)h(x) + s(x)$ with $s(x) = c_0 + c_1 x + \cdots + c_t x^t$, $t \leq r-1$ by the division algorithm. And we have

$$\begin{aligned} b_0 1_G + b_1\theta + \cdots, + b_m\theta^m &= g(\theta) = s(\theta) \\ &= c_0 1_G + c_1\theta + \cdots, + c_t\theta^t. \end{aligned}$$

Therefore, $Cf_F(G)$ is generated by $\{1_G, \theta^1, \theta^2, \cdots, \theta^{r-1}\}$.

Hence $\{1_G, \theta^1, \theta^2, \cdots, \theta^{r-1}\}$ is an F-basis of $Cf_F(G)$.
Since $dim_F Cf_F(G) = k$, we have $r = k$. Therefore the minimal polynomial $q(x) = x^k + a_{k-1}x^{k-1} + \cdots, + a_0$ of θ is the characteristic polynomial of $T^*(\theta)$. Hence θ takes exactly k distinct values.

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THEOREM 2.3. *Let α and β be any ordered F-basis of $Cf_F(G)$. Then*

$$[T(\theta)]_\alpha^\alpha = \{(X_\alpha X_\beta^{-1})^t\}^{-1}[T(\theta)]_\beta^\beta (X_\alpha X_\beta^{-1})^t.$$

Proof. Let $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ and $\beta = \{\beta_1, \beta_2, \cdots, \beta_k\}$. Let $\alpha_i = \sum_{t=1}^{k} a_{it}\beta_t$. Then

$$[T(1_G)]_\alpha^\beta = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{k1} \\ a_{21} & a_{22} & \dots & a_{k2} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{1k} & a_{2k} & \dots & a_{kk} \end{pmatrix}$$

Therefore, $X_\alpha = \{[T(1_G)]_\alpha^\beta\}^t X_\beta$.

Of course, $X = \{[T(1_G)]_{Irr_F(G)}^\alpha\}^t X_\alpha$ for any ordered F-basis α. Since X and $[T(1_G)]_{Irr_F(G)}^\alpha$ are invertible, X_α is invertible. Thus

$$\begin{aligned} &X^{-1}M(\theta)^t X \\ &= X_\alpha^{-1}\{[T(1_G)]_\alpha^{Irr_F(G)}\}^t\{[T(\theta)]_{Irr_F(G)}^{Irr_F(G)}\}^t\{[T(1_G)]_{Irr_F(G)}^\alpha\}^t X_\alpha \\ &= X_\alpha^{-1}\{[T(\theta)]_\alpha^\alpha\}^t X_\alpha. \end{aligned}$$

Since $X^{-1}M(\theta)^t X = X_\beta^{-1}\{[T(\theta)]_\beta^\beta\}^t X_\beta$ for ordered F-basis β,

$$X_\alpha^{-1}\{[T(\theta)]_\alpha^\alpha\}^t X_\alpha = X^{-1}M(\theta)^t X = X_\beta^{-1}\{[T(\theta)]_\beta^\beta\}^t X_\beta.$$

Hence $[T(\theta)]_\alpha^\alpha = \{(X_\alpha X_\beta^{-1})^t\}^{-1}[T(\theta)]_\beta^\beta (X_\alpha X_\beta^{-1})^t$.

THEOREM 2.4. *Assume that G has exactly k distinct conjugacy classes. If $Cf_F(G)$ is generated by $\{\theta^i | i \geq 0\}$ for some $\theta \in Cf_F(G)$, then the followings hold.*

(1) *$\{1_G, \theta^1, \theta^2, \cdots, \theta^{k-1}\}$ is an F-basis of $Cf_F(G)$.*

(2) *θ takes exactly k distinct values.*

Proof. Let T and T^* be representation defined in the introduction. Let $q(x) = x^r + a_{r-1}x^{r-1} + \cdots, + a_0$ be the minimal polymonial of $T^*(\theta)$. Then $T^*(\theta^r + a_{r-1}\theta^{r-1} + \cdots, + a_0 1_G) = 0$ iff $\theta^r + a_{r-1}\theta^{r-1} + \cdots, + a_0 1_G = 0$. Therefore, $q(x) = x^r + a_{r-1}x^{r-1} +$