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COMPLETE CONVERGENCE FOR WEIGHTED SUMS OF ARRAYS OF RANDOM ELEMENTS

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1. Introduction

Let $(B, \| \ \|)$ be a real separable Banach space. Let $(\Omega, \mathcal{F}, P)$ denote a probability space. A random elements in B is a function from Ω into B which is $\mathcal{F}$-measurable with respect to the Borel σ-field $\mathcal{B}(B)$ in B. An array $\{X_{nk}\}$ of random elements in B is said to be uniformly bounded by a random variable X if for all n and k and for each $t > 0$,

$$P(\|X_{nk}\| > t) \le P(|X| > t).$$

A separable Banach space B is said to be of type $p, 1 \le p \le 2$, if there exists a constant C such that

$$E\Big\|\sum_{k=1}^{n} X_k\Big\|^p \le C \sum_{k=1}^{n} E\|X_k\|^p$$

for all independent random elements $X_1, \cdots, X_n$ in B with mean zero and finite p-th moments. A sequence $\{U_n\}$ of random elements in B is said to converge completely to zero if for each $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P(\|U_n\| > \epsilon) < \infty.$$

Note that complete convergence implies almost surely convergence by Borel Cantelli lemma.

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Pruitt[9] investigated the complete convergence of

$$\sum_{k=1}^{\infty} a_{nk} X_k$$

when $\{X_n\}$ are i.i.d. random variables, and $\{a_{nk}, n \geq 1, k \geq 1\}$ is a Toeplitz array. Recall that an array $\{a_{nk}\}$ is a Toeplitz if

(i) $\lim_{n\to\infty} a_{nk} = 0$ for all k, and
(ii) $\sum_{k=1}^{\infty} |a_{nk}| \leq C$ for all n.

Generalizing a result of Pruitt, Rohatgi[10] proved the following result.

THEOREM 1.1. (Rohatgi[10]) *Let $\{X_n\}$ be a sequence of independent random variables with $EX_n = 0$ for all n which is uniformly bounded by a random variable X with $E|X|^{1+1/r} < \infty$ for some $r > 0$. Let $\{a_{nk}\}$ be a Toeplitz array satisfying* $\max_k |a_{nk}| = O(1/n^r)$. *Then $\sum_{k=1}^{\infty} a_{nk} X_k \to 0$ completely.*

Padgett and Taylor[7] considered the problem of extending of Rohatgi's theorem for real-valued random variables to Banach-valued random elements. They noted that Rohatgi's theorem can not be extended directly to separable Banach spaces. Wang and Rao[16] extended Rohatgi's result to the uniformly tight random elements. Taylor[14] obtained complete convergence for rowwise independent, uniformly bounded random elements in B-convex space. As a corollary of this result, he obtained a version of Rohatgi's result in B-convex space. The main theorem of Taylor is as follows.

THEOREM 1.2. (Taylor[14]) *Let $\{X_{nk}\}$ be an array of rowwise independent random elements in a separable B-convex space B with $EX_{nk} = 0$ for all n and k. Let $\{a_{nk}\}$ be a Toeplitz array satisfying* $\max_k |a_{nk}| = O(1/n^r)$ *for some $r > 0$. If $\{X_{nk}\}$ is uniformly bounded by a random variable X with $E|X|^{1+1/r} < \infty$, then $\sum_{k=1}^{\infty} a_{nk} X_{nk} \to 0$ completely.*

The purpose of this paper is to extend Taylor's theorem to general Banach spaces.

The convergence of the form

$$\sum_{k=1}^{n} a_{nk} X_{nk} \to 0 \text{ completely} \tag{1.1}$$

can be founded in Bozorgnia, Patterson, and Taylor[2],Sung[12], Taylor and Hu[15], and Wang, Rao and Yang[17], where $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ is an array of rowwise independent random elements, and $\{a_{nk}, 1 \le k \le n, n \ge 1\}$ is an array of constants. It should be noted that classical limit theorems for independent random variables hold for random elements under the additional condition of convergence in probability; see [1], [3], [4], [5], [6], [11], [12], and [17]. For example, Wang, Rao, and Yang[17] showed that (1.1) holds if and only if

$$\sum_{k=1}^{n} a_{nk} X_{nk} \to 0 \text{ in probability,} \tag{1.2}$$

when $a_{nk} = 1/n^{1/p}, 1 \le k \le n$, for some $1 \le p < 2$, and $\{X_{nk}\}$ is rowwise independent, uniformly bounded by a random variable X with $E|X|^{2p} < \infty$.

In this paper, we show that $\sum_{k=1}^{\infty} a_{nk} X_{nk} \to 0$ completely if and only if $\sum_{k=1}^{\infty} a_{nk} X_{nk} \to 0$ in probability under Taylor's conditions without B-convexity. We also obtain the equivalence of $\sum_{k=1}^{\infty} a_{nk} X_{nk} \to 0$ in probability and in L^1 under some restrictions on $\{a_{nk}\}$ and $\{X_{nk}\}$.

Throughout this paper, C will always stand for a positive constant which may be different in various places.

2. Main Results

To prove theorem 2.2, we need the following lemma.

LEMMA 2.1. (de Acosta[1]) *Let $X_1, \cdots, X_n$ be independent random elements in B with $E\|X_k\|^r < \infty$ for $k = 1, \cdots, n$ and $1 \le r \le 2$. Then*

$$E\big|\, \|S_n\| - E\|S_n\| \,\big|^r \le C_r \sum_{k=1}^{n} E\|X_k\|^r,$$

where $S_n = \sum_{k=1}^{n} X_k$, and C_r is a positive constant depending only on r.

The next theorem, our first main theorem, shows the equivalence of $\sum_{k=1}^{\infty} a_{nk} X_{nk} \to 0$ in probability and in L^1 under some restrictions on $\{a_{nk}\}$ and $\{X_{nk}\}$.

THEOREM 2.2. *Let $\{X_{nk}\}$ be an array of rowwise independent random elements in B such that $\max_{n,k} E\|X_{nk}\|^\alpha < \infty$ for some $\alpha > 1$. Let $\{a_{nk}\}$ be a Toeplitz array satisfying $\max_k |a_{nk}| = O(1/n^r)$ for some $r > 0$. Then the following statements are equivalent.*

(i) $\sum_{k=1}^\infty a_{nk}X_{nk} \to 0$ *in probability as* $n \to \infty$.
(ii) $\sum_{k=1}^\infty a_{nk}X_{nk} \to 0$ *in* L^1 *as* $n \to \infty$.

Proof. Since (ii)$\Rightarrow$ (i) is obvious, we will show (i)$\Rightarrow$(ii). Assume that $\sum_{k=1}^\infty a_{nk}X_{nk} \to 0$ in probability as $n \to \infty$. Since $\sum_{k=1}^\infty |a_{nk}|$ is bounded, we can choose a sequence $\{r_n\}$ of integers such that

$$\sum_{k=r_n+1}^\infty |a_{nk}| \le \frac{1}{n^2}, \quad \text{for each } n.$$

Then

$$E\Big\|\sum_{k=r_n+1}^\infty a_{nk}X_{nk}\Big\| \le \sum_{k=r_n+1}^\infty |a_{nk}|E\|X_{nk}\| \le C\frac{1}{n^2} \to 0, \tag{2.1}$$

since $\max_{n,k} E\|X_{nk}\|$ is finite. Hence it is enough to show that

$$\sum_{k=1}^{r_n} a_{nk}X_{nk} \to 0 \text{ in } L^1. \tag{2.2}$$

Let $\beta = \min\{2, \alpha\}$. From Lemma 2.1,

$$\begin{aligned} &E\Big|\ \Big\|\sum_{k=1}^{r_n} a_{nk}X_{nk}\Big\| - E\Big\|\sum_{k=1}^{r_n} a_{nk}X_{nk}\Big\|\ \Big|^\beta \\ \le& C\sum_{k=1}^{r_n} E\|a_{nk}X_{nk}\|^\beta \le C\max_{n,k} E\|X_{nk}\|^\beta \sum_{k=1}^\infty |a_{nk}|^\beta \\ \le& C\max_{n,k} E\|X_{nk}\|^\beta \Big(\frac{1}{n^r}\Big)^{\beta-1} \sum_{k=1}^\infty |a_{nk}| \to 0, \end{aligned}$$

since $\beta > 1$. Hence

$$\Big\|\sum_{k=1}^{r_n} a_{nk}X_{nk}\Big\| - E\Big\|\sum_{k=1}^{r_n} a_{nk}X_{nk}\Big\| \to 0 \ \text{ in probability.} \tag{2.3}$$

Note that

$$\left\|\sum_{k=1}^{r_n} a_{nk}X_{nk}\right\| \to 0 \text{ in probability,}$$

by (i) and (2.1). Combining this result and (2.3) gives the desired result (2.2).

The following lemma plays an essential role in our second main result.

LEMMA 2.3. (Sung[12]) *Let $X_1, \cdots, X_n$ be independent random elements in B such that*

$$\|X_k\| \le b_k, \quad 1 \le k \le n,$$

and let $S_n = \sum_{k=1}^{n} X_k$. Then, for any $t > 0$

$$E\left[\exp(t\|S_n\|)\right] \le \exp\left\{tE\|S_n\| + 2t^2 \sum_{k=1}^{n} e^{2tb_k} E\|X_k\|^2\right\}.$$

The next theorem extends Rohatgi's result to separable Banach space. It is also a generalization of Taylor's theorem [14].

THEOREM 2.4. *Let $\{X_{nk}\}$ be an array of rowwise independent random elements in a separable Banach space B which is uniformly bounded by a random variable X with $E|X|^{1+1/r} < \infty$ for some $r > 0$. If Toeplitz array $\{a_{nk}\}$ satisfies $\max_k |a_{nk}| = O(1/n^r)$, then the following statements are equivalent.*

(i) $\sum_{k=1}^{\infty} a_{nk}X_{nk} \to 0$ *in probability as* $n \to \infty$.
(ii) $\sum_{k=1}^{\infty} a_{nk}X_{nk} \to 0$ *completely as* $n \to \infty$.

We need only to prove (i)$\Rightarrow$(ii). The proof is completed by the following three lemmas, since

$$\begin{aligned}
&\left\{\left\|\sum_{k=1}^{\infty} a_{nk}X_{nk}\right\| \ge \epsilon\right\} \\
&\subset \left\{\sum_{k=1}^{\infty} a_{nk}X_{nk}I(\|a_{nk}X_{nk}\| < n^{-\alpha})\| \ge \frac{\epsilon}{2}\right\} \\
&\cup \left\{\|a_{nk}X_{nk}\| \ge \frac{\epsilon}{2} \text{ for some } k\right\} \\
&\cup \left\{\|a_{nk}X_{nk}\| \ge n^{-\alpha} \text{ for at least two values of } k\right\}.
\end{aligned}$$

The following Lemma 2.5 and Lemma 2.6 are in Taylor[14](or see Rohatgi[10]). The proof of Lemma 2.7 is different from that of Lemma 3 in Taylor[14]. It seems that the proof of Lemma 3 in Taylor for B-convex Banach space can not be adapted to general Banach spaces.

LEMMA 2.5. *If* $E|X|^{1+1/r} < \infty$ *and* $\max_k |a_{nk}| = O(1/n^r)$, *then for every* $\epsilon > 0$

$$\sum_{n=1}^{\infty} P(\|a_{nk}X_{nk}\| \geq \epsilon \text{ for some } k) < \infty.$$

LEMMA 2.6. *If* $E|X|^{1+1/r} < \infty$ *and* $\max_k |a_{nk}| = O(1/n^r)$, *then for* $\alpha < r/(2r+2)$

$$\sum_{n=1}^{\infty} P(\|a_{nk}X_{nk}\| \geq n^{-\alpha} \text{ for at least two values of } k) < \infty.$$

LEMMA 2.7. *If* $E|X|^{1+1/r} < \infty$ *and* $\max_k |a_{nk}| = O(1/n^r)$, *then*

$$\sum_{k=1}^{\infty} a_{nk}X_{nk} \to 0 \text{ in probability}$$

implies

$$\sum_{n=1}^{\infty} P\Big(\|\sum_{k=1}^{\infty} a_{nk}X_{nk}I(\|a_{nk}X_{nk}\| < n^{-\alpha})5\| \geq \epsilon\Big) < \infty,$$

where $0 < \alpha < r$.

Proof. Define $Z_{nk} = X_{nk}I_(\|a_{nk}X_{nk}\| < n^{-\alpha})$. Then $E\|Z_{nk}\| \leq E|X|$. Let $\{r_n\}$ be as in the proof of Theorem 2.2. By Markov inequality

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\Big(\|\sum_{k=1}^{\infty} a_{nk}Z_{nk}\| \geq \epsilon\Big) \\
&\leq \sum_{n=1}^{\infty} P\Big(\|\sum_{k=1}^{r_n} a_{nk}Z_{nk}\| \geq \frac{\epsilon}{2}\Big) + \sum_{n=1}^{\infty} P\Big(\|\sum_{k=r_n+1}^{\infty} a_{nk}Z_{nk}\| \geq \frac{\epsilon}{2}\Big) \\
&\leq \sum_{n=1}^{\infty} P\Big(\|\sum_{k=1}^{r_n} a_{nk}Z_{nk}\| \geq \frac{\epsilon}{2}\Big) + \frac{2}{\epsilon}\sum_{n=1}^{\infty}\sum_{k=r_n+1}^{\infty} E\|a_{nk}Z_{nk}\| \\
&\leq \sum_{n=1}^{\infty} P\Big(\|\sum_{k=1}^{r_n} a_{nk}Z_{nk}\| \geq \frac{\epsilon}{2}\Big) + \frac{2}{\epsilon}E|X|\sum_{n=1}^{\infty}\frac{1}{n^2}.
\end{aligned}$$

Hence it is enough to show that

$$\sum_{n=1}^{\infty} P\left(\|\sum_{k=1}^{r_n} a_{nk} Z_{nk}\| > \frac{\epsilon}{2}\right) < \infty. \tag{2.4}$$

Fix $n \geq 1$. Let $t = 4\log n/\epsilon$. Since $\|a_{nk}Z_{nk}\| < n^{-\alpha}$, it follows by Markov inequality and Lemma 2.3 that

$$\begin{aligned} &P\Big(\|\sum_{k=1}^{r_n} a_{nk}Z_{nk}\| > \frac{\epsilon}{2}\Big) \\ &\leq \exp\Big\{-\frac{\epsilon t}{2}\Big\} E\left[exp(t\|\sum_{k=1}^{r_n} a_{nk}Z_{nk}\|)\right] \\ &\leq \exp\left\{-\frac{\epsilon t}{2} + tE\|\sum_{k=1}^{r_n} a_{nk}Z_{nk}\| + 2t^2 e^{2tn^{-\alpha}} \sum_{k=1}^{r_n} E\|a_{nk}Z_{nk}\|^2\right\}. \end{aligned} \tag{2.5}$$

Now we calculate the power of exp in the last expression of (2.5). From triangular inequality

$$\begin{aligned} &E\|\sum_{k=1}^{r_n} a_{nk}Z_{nk}\| \\ &= E\|\sum_{k=1}^{r_n} a_{nk}(X_{nk} - X_{nk}I(\|a_{nk}X_{nk} \geq n^{-\alpha}))\| \\ &\leq E\|\sum_{k=1}^{r_n} a_{nk}X_{nk}\| + E\|\sum_{k=1}^{r_n} a_{nk}X_{nk}I(\|a_{nk}X_{nk} \geq n^{-\alpha})\| \\ &\leq E\|\sum_{k=1}^{\infty} a_{nk}X_{nk}\| + E\|\sum_{k=r_n+1}^{\infty} a_{nk}X_{nk}\| \\ &\quad + E\|\sum_{k=1}^{r_n} a_{nk}X_{nk}I(\|a_{nk}X_{nk} \geq n^{-\alpha})\| \\ &\leq E\|\sum_{k=1}^{\infty} a_{nk}X_{nk}\| + \frac{E|X|}{n^2} + E\|\sum_{k=1}^{r_n} a_{nk}X_{nk}I(\|a_{nk}X_{nk} \geq n^{-\alpha})\| \to 0, \end{aligned}$$

since the first term in the last expression converges to 0 by Theorem 2.2, the second term converges to 0 clearly, and the third term converges to 0 by the following fact.

$$\begin{aligned} &E\|\sum_{k=1}^{r_n} a_{nk}X_{nk}I(\|a_{nk}X_{nk} \geq n^{-\alpha})\| \\ \leq &\sum_{k=1}^{r_n} |a_{nk}|E\|X_{nk}\|I(\|X_{nk}\| \geq Cn^{r-\alpha}) \\ \leq &E|X|I(|X| > Cn^{r-\alpha})\sum_{k=1}^{\infty}|a_{nk}| \\ \leq &CE|X|I(|X| > Cn^{r-\alpha}) \to 0. \end{aligned}$$

Since $E\|a_{nk}Z_{nk}\| < n^{-\alpha}$, we obtain

$$\sum_{k=1}^{r_n} E\|a_{nk}Z_{nk}\|^2 \leq \frac{1}{n^\alpha}\sum_{k=1}^{r_n} E\|a_{nk}Z_{nk}\| \leq \frac{E|X|}{n^\alpha}\sum_{k=1}^{\infty}|a_{nk}| \leq C\frac{E|X|}{n^\alpha}.$$

Thus, for any fixed $\eta > 0$, the power of exp in the last expression of (2.5) is bounded by

$$-2\log n + \eta\log n + C\frac{\log^2 n}{n^\alpha}e^{C\log n/n^\alpha}$$

for all n sufficiently large. Choose η such that $-2+\eta < -1$. Then

$$\begin{aligned} &\sum_{n=1}^{\infty}\exp\left\{-2\log n + \eta\log n + C\frac{\log^2 n}{n^\alpha}e^{C\log n/n^\alpha}\right\} \\ \leq &C\sum_{n=1}^{\infty}\exp\{-2\log n + \eta\log n\} < \infty. \end{aligned}$$

Therefore the desired result (2.4) holds.

In Theorem 2.4, the convergence in probability is obtained by imposing an additional geometric condition on B.

COROLLARY 2.8. (Taylor[14]) *Let* $\{X_{nk}\}$ *and* $\{a_{nk}\}$ *be as in Theorem 2.4. If* $EX_{nk} = 0$ *and Banach space is B-convex, then*

$$\sum_{k=1}^{\infty} a_{nk}X_{nk} \to 0 \text{ completely.}$$

Proof. By Theorem 2.4, it suffices to show that

$$\sum_{k=1}^{\infty} a_{nk}X_{nk} \to 0 \text{ in probability.}$$

Note that B-convexity implies that B is of type p for some $p > 1$, see Pisier[8] (or Taylor[14]). Let $\beta = \min\{1 + 1/r, p\}$. Since Banach space B is of type β,

$$\begin{aligned} E\|\sum_{k=1}^{\infty} a_{nk}X_{nk}\|^{\beta} \leq & C\sum_{k=1}^{\infty} E\|a_{nk}X_{nk}\|^{\beta} \\ \leq & CE|X|^{\beta}\Big(\frac{1}{n^r}\Big)^{\beta-1}\sum_{k=1}^{\infty}|a_{nk}| \to 0. \end{aligned}$$

Thus the proof is complete.

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