J. Korean Math. Soc. **32** (1995), No. 4, pp. 661–678

ESTIMATES OF INVARIANT METRICS ON SOME PSEUDOCONVEX DOMAINS IN $\mathbb{C}^n$

SANGHYUN CHO

1. Introduction

In this paper we will estimate from above and below the values of the Bergman, Caratheodory and Kobayashi metrics for a vector X at z, where z is any point near a given point z_0 in the boundary of pseudoconvex domains in $\mathbb{C}^n$. Throughout this paper, Ω will be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth defining function r and $z_0 \in b\Omega$ is a point of finite type m in the sense of D'Angelo [7], and the Levi-form $\partial\bar{\partial}r(z)$ of $b\Omega$ has $(n-2)$-positive eigenvalues at z_0. Note that the type m at z_0 is an even integer in this case. We first give the definition of each of the above metrics. Let X be a holomorphic tangent vector at a point z in Ω. Denote the set of holomorphic functions on Ω by $A(\Omega)$. Then the Bergman metric $B_\Omega(z;X)$, the Caratheodory metric $C_\Omega(z;X)$ and the Kobayashi metric $K_\Omega(z;X)$ are defined by

$$C_\Omega(z;X) = \sup\{|Xf(z)| : f \in A(\Omega),\ \ \|f\|_{L^\infty\Omega} \le 1\}$$

$$K_\Omega(z;X) = \inf\{1/r : \exists f : D_r \subset \mathbb{C}^1 \to \mathbb{C}^n \text{ such that } f_*\Big(\frac{\partial}{\partial z}|_0\Big) = X\}$$

$$B_\Omega(z : X) = b_\Omega(z;X)/(K_\Omega(z,\bar{z}))^{\frac{1}{2}},$$

where D_r denotes the disc of radius r in $\mathbb{C}^1$, and

$$K_\Omega(z,\bar{z}) = \sup\{|f(z)|^2 : f \in A(\Omega),\ \ \|f\|_{L^2(\Omega)} \le 1\}$$
$$b_\Omega(z;X) = \sup\{|Xf(z)| : f \in A(\Omega),\ \ f(z)=0,\ \ \|f\|_{L^2(\Omega)} \le 1\}.$$

Received August 25, 1994.

1991 AMS Subject Classification: 32F15.

Key words: Invariant metrics, Bergman kernel function, finite 1-type, plurisubharmonic functions.

This paper was partially supported by Non directed research fund, K.R.F.,1993 and GARC-KOSEF fund, 1994.

We may assume that $\partial r/\partial z_1 \neq 0$ in a small neighborhood U of z_0. After a linear change of coordinates, we can find coordinate functions $z_1, z_2, \ldots, z_n$ defined on U such that

$$\begin{aligned} L_1 &= \frac{\partial}{\partial z_1}, \\ L_j &= \frac{\partial}{\partial z_j} + b_j \frac{\partial}{\partial z_1}, L_j r \equiv 0, \quad b_j(z_0) = 0, \quad j = 2, \ldots, n, \end{aligned}$$

which form a basis of $\mathbb{C}T(U)$ and satisfy

$$\partial\bar{\partial} r(z_0)(L_i, \bar{L}_j) = \delta_{ij}, \quad 2 \leq i, j \leq n-1,$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. For any integers $j, k > 0$, set

$$\mathcal{L}_{j,k}\partial\bar{\partial} r(z) = \underbrace{L_n \ldots L_n}_{(j-1)\text{ times}} \underbrace{\bar{L}_n \ldots \bar{L}_n}_{(k-1)\text{ times}} \partial\bar{\partial} r(z)(L_n, \bar{L}_n),$$

and define

$$\begin{aligned} C_l(z) &= \max\{|\mathcal{L}_{j,k}\partial\bar{\partial} r(z)|; j + k = l\}, \\ \eta(z, \delta) &= \min\{(\delta/C_l(z))^{1/l} : l = 2, \ldots, m\}. \end{aligned} \tag{1.1}$$

Let $X = b_1 L_1 + b_2 L_2 + \ldots + b_n L_n$ be a holomorphic tangent vector at z and set

$$\begin{aligned} M_m(z; X) = &|b_1||r(z)|^{-1} + \sum_{k=2}^{n-1} |b_k||r(z)|^{-1/2} \\ &+ |b_n| \sum_{l=2}^{m} |C_l(z)|^{1/l} |r(z)|^{-1/l}. \end{aligned} \tag{1.2}$$

Then we can state the main result as follows

THEOREM 1. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ and let $z_0 \in b\Omega$ be a point of finite type m in the sense of D'Angelo. Also assume that the Levi-form $\partial\bar{\partial} r(z)$ of $b\Omega$ has $(n-2)$-positive eigenvalues at z_0. Then there exist a neighborhood U about z_0 and positive constants c and C such that for all $X = b_1 L_1 + \ldots b_n L_n$ at $z \in U \cap \Omega$,*

$$cM_m(z; X) \leq B_\Omega(z; X), \quad C_\Omega(z; X), \quad K_\Omega(z : X) \leq CM_m(z; X). \tag{1.3}$$

REMARK. Because $|C_m(z)| \geq c' > 0$ for all $z \in U \cap \Omega$, (1.2) says, in particular, that

$$B_\Omega(z;X), C_\Omega(z;X), K_\Omega(z;X) \gtrsim (|b_1||r(z)|^{-1} + \sum_{k=2}^{n-1} |b_k||r(z)|^{-1/2} + |b_n||r(z)|^{-1/m}$$

for a holomorphic vector field $X = b_1 L_1 + \ldots + b_n L_n$ at z.

Several authors found some results about these metrics for some pseudoconvex domains in $\mathbb{C}^n$, but in each case the lower bounds are different from the upper bounds [1,5,8,9,12]. In [2], Catlin got a result similar to above theorem in $\mathbb{C}^2$, and this has motiviated the author to investigate the above theorem. To prove the theorem, we must get a complete geometric analysis near z_0 and this will be done by using the "maximal plurisubharmonic functions" constructed in [6]. In [10], K.T. Hahn got the following inequalities

$$(1.4) \qquad C_\Omega(z;X) \leq B_\Omega(z;X), \quad K_\Omega(z;X).$$

Therefore the estimates for the lower bounds of $C_\Omega(z;X)$ will suffice for the lower bounds of $B_\Omega(z;X)$ and $K_\Omega(z;X)$. For upper bounds of $B_\Omega(z;X)$, we will use the following estimates for the Bergman kernel function $K_\Omega(z,\bar{z})$

$$(1.5) \qquad K_\Omega(z,\bar{z}) \approx \sum_{l=2}^{m} |C_l(z)|^{2/l} |r(z)|^{-n-2/l},$$

which was shown by the author in [6].

Although we are employing some of the methods similar to those of Catlin in $\mathbb{C}^2$-case, where he used estimates for the $\bar{\partial}$-Neumann operator. we will show some technical theorems in detail to clarify the difference between $\mathbb{C}^2$ and $\mathbb{C}^n$ case.

2. Pushing out the boundary and Bumping theorem

For each $z' \in U$, we take the biholomorphism $\Phi_{z'}^{-1} \ : \ \mathbb{C}^n \to \mathbb{C}^n$ which straightens $b\Omega$ near z_0 [6, Proposition 2.2]. That is, $\Phi_{z'}$ satisfies $\Phi_{z'}^{-1}(z') = 0, \ \Phi_{z'}^{-1}(z) = \zeta$, and

$$
\begin{aligned}
r(\Phi_{z'}(\zeta)) =& r(z') + Re\zeta_1 + \sum_{\alpha=2}^{n-1} \sum_{\substack{j+k\leq m/2 \\ j,k>0}} Re\left(b^{\alpha}_{j,k}(z')\zeta_n^j\bar{\zeta}_n^k\zeta_\alpha\right) \\
&+ \sum_{\substack{j+k\leq m \\ j,k>0}} a_{j,k}(z')\zeta_n^j\bar{\zeta}_n^k + \sum_{\alpha=2}^{n-1}|\zeta_\alpha|^2 \\
&+ \mathcal{O}\left(|\zeta_1||\zeta| + |\zeta''|^2|\zeta| + |z''||\zeta_n|^{m/2+1} + |\zeta_n|^{m+1}\right).
\end{aligned} \tag{2.1}
$$

Set $\rho(\zeta) = r \circ \Phi_{z'}(\zeta)$, and set

(2.2)

$$
\begin{aligned}
A_l(z') &= \max\{|a_{j,k}(z')|; j+k=l\}, \quad 2 \leq l \leq m, \\
B_{l'}(z') &= \max\{|b^{\alpha}_{j,k}(z')|; j+k=l', \quad 2\leq \alpha \leq n-1,\}, \quad 2 \leq l' \leq m/2.
\end{aligned}
$$

For each $\delta > 0$, we define $\tau(z', \delta)$ as follwos;

$$
\tau(z',\delta) = \min_{\substack{2\leq l\leq m \\ 2\leq l'\leq m/2}} \left\{(\delta/A_l(z'))^{1/l}, \ (\delta^{1/2}/B_{l'}(z'))^{1/l'}\right\}. \tag{2.3}
$$

Since $A_m(z_0) \geq c > 0$, it follows that $A_m(z') \geq c' > 0$ for all $z' \in U$ if U is sufficiently small. This gives the inequality,

$$
\delta^{1/2} \lesssim \tau(z',\delta) \lesssim \delta^{1/m}, \quad z' \in U.
$$

REMARK 2.1. It was shown in [6, section 2] that $(\delta^{1/2}z'(B_l'(z')) \gg \tau(z',\delta)$ whenever $\delta > 0$ is sufficiently small. Hence the terms mixed with ζ_n and ζ_α, $\alpha = 2, \ldots, n-1$, would not be and important ones in (2.1) and (2.3) and hence $\tau(z',\delta) = \min\left\{(\delta/A_l(z'))^{1/l}; \ 2 \leq l \leq m\right\}$ for sufficiently small δ.

The definition of $\tau(z',\delta)$ easily implies that if $\delta' < \delta''$, then

$$
(\delta'/\delta'')^{1/2}\tau(z',\delta'') \leq \tau(z',\delta') \leq (\delta'/\delta'')^{1/m}\tau(z',\delta'').
$$

Because we are fixing z' in this section, we set $\tau_1 = \delta, \tau_2 = \ldots = \tau_{n-1} = \delta^{1/2}$, $\tau_n = \tau(z', \delta) = \tau$ and define

$$R_\delta(z') = \{\zeta \in \mathbb{C}^n; |\zeta_k| < \tau_k,\ k = 1, 2, \ldots, n\}, \text{ and}$$
$$Q_\delta(z') = \{\Phi_{z'}(\zeta); \zeta \in R_\delta(z')\}.$$

Then for $z \in Q_\delta(z')$, the author showed in [6, Proposition 2.7] that

$$\tau(z', \delta) \lesssim \eta(z, \delta) \lesssim \tau(z', \delta) \text{ and} \tag{2.4}$$
$$\eta(z, \delta) \approx \tau(z, \delta).$$

For $\epsilon > 0$, we let $\Omega_\epsilon = \{z : r(z) < \epsilon\}$ and set $S(\epsilon) = \{z : -\epsilon < r(z) < \epsilon\}$. In [6, Proposition 3.2], the author proved the following theorem which shows the existence of smooth plurisubharmonic functions on $\overline{\Omega}$ with "maximal Hessian" near $b\Omega$.

THEOREM 2.1. *For all small $\delta > 0$, there is a plurisubharmonic function $\lambda_\delta \in C^\infty(\Omega_\delta)$ with the following properties*

(i) $|\lambda_\delta(z)| \leq 1$, $z \in U \cap \Omega_\delta$.

(ii) *For all $L = \sum_{j=1}^n b_j L_j$ at $z \in U \cap S(\delta)$,*

$$\partial\overline{\partial}\lambda_\delta(z)(L, \overline{L}) \approx \delta^{-2}|b_1|^2 + \delta^{-1}\sum_{k=2}^{n-1}|b_k|^2 + \tau^{-2}|b_n|^2,$$

(iii) *If $\Phi_{z'}$ is the map associted with a given $z' \in U \cap S(\delta)$, then for all $\zeta \in R_\delta(z')$ with $|\rho(\zeta)| < \delta$,*

$$|D^\alpha(\lambda_\delta \circ \Phi_{z'})(\zeta)| \lesssim C_\alpha \delta^{-\alpha_n}\delta^{-1/2(\alpha_2+\ldots+\alpha_{n-1})}\tau^{-\alpha_n}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$.

With this family of functions λ_δ, we shall construct for each $z' \in U \cap b\Omega$ and each small $\delta > 0$, a domain (locally defined in U) $\Omega_{z',\delta}$ which contains Ω such that the boundary of $\Omega_{z',\delta}$ is pushed out as far as possible, given the constraints that $d(z', b\Omega_{z',\delta}) < \delta$ and that $b\Omega_{z',\delta}$ is pseudoconvex. Since z' will be fixed in this section, we will work in ζ-coordinates defined by $\Phi_{z'}(\zeta) = z$.

Set $\rho(\zeta) = r(\Phi_{z'}(\zeta))$ and set $U' = \{\zeta : \Phi_{z'}(\zeta) \in U\}$. For all small s and $\delta > 0$, define

$$J_\delta(z',\zeta) = \left[\delta^2 + |\zeta_1|^2 + \sum_{\alpha=2}^{n-1} |\zeta_\alpha|^4 + \sum_{l=2}^{m} A_l(z')^2 |\zeta_n|^{2l}\right]^{1/2} \tag{2.5}$$

and

$$W_{s,\delta}(z') = \{\zeta \in U' : |\rho(\zeta)| < sJ_\delta(\zeta)\}. \tag{2.6}$$

NOTE. From Remark 2.1, and (2.3), the terms $B_{l'}(z')^2|\zeta_n|^{2l'}|\zeta_\alpha|^2$, $2 \le l' \le m/2$ will be absorbed into $\sum_{l=2}^m A_l(z')^2|\zeta_n|^{2l}$ in the definition of $J_\delta(z',\zeta)$.

Set $J_\delta(z',\zeta) = J_\delta(\zeta)$ for the convenience.

PROPOSITION 2.2. *For each $z' \in U \cap b\Omega$ and each small $\delta > 0$, there exists a small real-valued function $H_{z',\delta}(\zeta)$ defined in $W_{s,\delta}(z')$ (where s is a small constant independent of z' and δ) such that*

(i) $-J_\delta(\zeta) \approx H_{z',\delta}(\zeta)$,

(ii) *for any* $L = b_1L_1' + b_2L_2' + \ldots + b_nL_n'$,

$$\partial\bar\partial H_{z',\delta}(L,\overline{L})(\zeta) \approx J_\delta(\zeta)\left[\frac{|b_1|^2}{(J_\delta(\zeta))^2} + \sum_{k=2}^{n-1}\frac{|b_k|^2}{(J_\delta(\zeta))} + \frac{|b_n|^2}{\tau(z',J_\delta(\zeta))^2}\right],$$

(iii) *for any* $L = b_1L_1' + \ldots + b_nL_n'$ *at* ζ,

$$|LH_{z',\delta}| \lesssim J_\delta(\zeta)\left(\frac{|b_1|}{J_\delta(\zeta)} + \sum_{k=2}^{n1}\frac{|b_k|}{(J_\delta(\zeta))^{1/2}} + \frac{|b_n|}{\tau(z'.J_\delta(\zeta))}\right)$$

where $L_k' = (\Phi_{z'}^{-1})L_k$, $k = 1, 2, \ldots, n$.

Proof. Set $N_1 = [log_2(1/\delta)]$. Let $D_R = \{\zeta \in \mathbb{C}^n : |\zeta_i| < R, \quad i = 1, 2, \ldots, n\}$, and let $\psi \in C_0^\infty(D_2 - D_{1/4})$ be a function that satisfies $\psi(\zeta) = 1$ for $\zeta \in D_1 - D_{1/2}$. For all k, $1 \le k < N_1$, set

$$\psi_k(\zeta) = \psi\left(2^k\zeta_1, 2^{k/2}\zeta_2, \ldots, 2^{k/}\zeta_{n-1}, \tau(z', 2^{-k})^{-1}\zeta_n\right),$$

and for $k = N_1$, set

$$\psi_{N_1}(\zeta) = \phi\left(2^{N_1}\zeta_1, 2^{N_1/2}\zeta_2, \ldots, 2^{N_1/2}\zeta_{n-1}, \tau(z', 2^{-N_1})^{-1}\zeta_n\right),$$

where $\phi \in C_0^\infty(D_2)$ satisfies $\phi(\zeta) = 1$ for $\zeta \in D_1$. If one combines (2.3), (2.5) nd the fact that $(\delta^{1/2}/B_{l'}(z'))^{1/l'} \gg \tau(z', \delta)$ for $l' = 2, \ldots, m/2$, one obtains that

$$J_\delta(\zeta) \approx 2^{-k}, \quad \zeta \in \operatorname{supp} \psi_k. \tag{2.7}$$

For each $\delta > 0$, set $\lambda'_\delta = \lambda_\delta \circ \Phi_{z'}$, where λ_δ is the plurisubharmonic function as in Theorem 2.1. Choose N_0 so that $\lambda_{2^{-k}t}$ is well-defined for all $\zeta \in \operatorname{supp} \psi_k$ whenever $k \geq N_0$, and set

$$H_{z',\delta}(\zeta) = \sum_{k=N_0}^{N_1} 2^{-k}\psi_k(\zeta)(\lambda'_{2^{-k}t}(\zeta) - 2).$$

Then $H_{z',\delta}$ is well-defined (fixed finite sum independent of z' and δ). From (2.5), (2.7) and from the fact that $H_{z',\delta}(\zeta) \approx -2^{-k}$ for $\zeta \in \operatorname{supp} \psi_k$, property (i) follows. Also the major part of the Hessian of $H_{z',\delta}$ will be $\partial\overline{\partial}\lambda_{2^{-k}t}(\zeta)$ and other error terms will be absorbed into $\partial\overline{\partial}\lambda_{2^{-k}t}(\zeta)$ for sufficiently small t. This fact together property (i)proves properties (ii) and (iii). □

Set $\Omega_{z'} = \Phi_{z'}^{-1}(\Omega)$ and set $\Omega_{z',\epsilon} = \{\zeta \in \mathbb{C}^n : \rho(\zeta) < \epsilon\}$.

PROPOSITION 2.3. *For all small $\delta > 0$, there exist a function $g_\delta(\zeta)$ and constants $b > 0$ and $C > 0$ so that*

(i) *g_δ is defined and plurisubharmonic on $\Omega_{z',b\delta}$,*
(ii) *supp $g_\delta \subset R_{C\delta}(z') \cap \Omega_{z',b\delta}$,*
(iii) *$|g_\delta(\zeta)| \leq 1$, $\zeta \in \Omega_{z',b\delta}$,*
(iv) *if $L = b_1L'_1 + \ldots + b_nL'_n$ at $\zeta \in R_{b\delta}(z')$, then*

$$\partial\overline{\partial}g_\delta(L, \overline{L})(\zeta) \gtrsim \delta^{-2}|b_1|^2 + \delta^{-1}\sum_{k=1}^{n-1}|_k|^2 + \tau(z', \delta)^{-2}|b_n|^2, \text{ and}$$

(v) $|D^\alpha g_\delta(\zeta)| \lesssim C_\alpha \delta^{-\alpha_1}\delta^{-1/2(\alpha_2+\ldots+\alpha_{n-1})}\tau^{-\alpha_n}$, $\zeta \in R_{C\delta}(z')$, *where* $\alpha = (\alpha_1, \dots, \alpha_n)$.

Proof. If we use Theorem 2.1, the proof of this theorem will be very close to that o $\mathbb{C}^2$-case of Proposition 4.2 in [2] and hence it will be omitted here.

With Proposition 2.2, we can prove the following theorem which shows that the boundary of $\Omega_{z'}$ can be pushed out essentially as far as possible.

THEOREM 2.4. *For each sufficiently small* $\delta > 0$, *there isone parameter family of "maximal pushed-out" pseudoconvex domains* $\{\Omega^{\epsilon}_{z',\delta}\}_{\epsilon>0}$ *which contain* $\Omega_{z'}$ *near the origin.*

Proof. Let U_1' be a small neighborhood of the origin with $U_1' \subset\subset U' = \Phi_{z'}^{-1}(U)$. Then one has $|dH_{z',\delta}(\zeta)| \lesssim 1$ for $\zeta \in W_{s,\delta}(z')$ by the property (iii) of Theorem 2.2. Hence for all small $\epsilon > 0$, the function

$$\rho^{\epsilon}_{z',\delta}(\zeta) = \rho(\zeta) + \epsilon H_{z',\delta}(\zeta)$$

satisfies $\frac{\partial \rho^{\epsilon}_{z',\delta}}{\partial \zeta_1} \neq 0$ in U_1' and therefore form a family of defining functions of hypersurfaces $\{\zeta : \rho^{\epsilon}_{z'}(\zeta) = 0\}$ in $W_{s,\delta}(z')$. For $\zeta' \in b\Omega_{z'} \cap U_1'$, let ζ'' be the unique projection of ζ' onto $\{\zeta : \rho^{\epsilon}_{z',\delta}(\zeta) = 0\} = b\Omega^{\epsilon}_{z',\delta}$. Suppose $L''\rho^{\epsilon}_{z'}(\zeta'') = 0$ with $|L''| = 1$. If one writes $L'' = eL_1' + s_2L_2' + \dots + s_nL_n' = eL_1' + T'$, then $L''\rho^{\epsilon}_{z'}(\zeta'') = 0$ implies that

$$\begin{aligned} & eL_1'(\rho + \epsilon H_{z',\delta})(\zeta'') + T'(\rho + \epsilon H_{z',\delta})(\zeta'') \\ &= e(L_1'\rho + \epsilon L_1' H_{z',\delta})(\zeta'') + \epsilon T' H_{z',\delta}(\zeta'') = 0, \end{aligned}$$

which shows that

$$\begin{aligned} |e| &\approx \epsilon |T' H_{z',\delta}(\zeta'')| \\ &\lesssim \epsilon J_\delta(\zeta'') \left(\sum_{k=2}^{n-1} |s_k| J_\delta(\zeta'')^{-1/2} + \tau(z', J_\delta(\zeta''))^{-1} |s_n| \right). \end{aligned}$$

Therefore we may assume that $|T'| \geq /2$ provided that $|\zeta''|$ is sufficiently small (i.e., U' is sufficiently small). Because $\partial\overline{\partial}\rho(T', \overline{T}')(\zeta'') \geq$

0, one has

$$\begin{aligned}
&\partial\overline{\partial}\rho(L'',\overline{L}'')(\zeta'') = \partial\overline{\partial}\rho(T'+\epsilon L_1,\overline{T'+\epsilon L_1})(\zeta'') \\
&= \partial\overline{\partial}\rho(T',\overline{T}')(\zeta'') + \mathcal{O}(\epsilon) \\
&\geq -\epsilon J_\delta(\zeta'')\left(\tau(z',J_\delta(\zeta''))^{-1}|s_n| + \sum_{k=2}^{n-1}|s_k|J_\delta(\zeta'')^{-1/2}\right).
\end{aligned}$$

Since $|\zeta''-\zeta'| \lesssim J_\delta(\zeta')$, one sees that $J_\delta(\zeta') \approx J_\delta(\zeta'')$. If one combines this fact and $|T'| \geq 1/2$, and the property (ii) of Proposition 2.2, one gets,

$$\begin{aligned}
&\partial\overline{\partial}\rho^{\epsilon}_{z',\delta}(L'',\overline{L}'') \\
&\geq -C\epsilon J_\delta(\zeta^{prime})\left(\tau(z',J_\delta(\zeta'))^{-1}|s_n| + \sum_{k=2}^{n-1}|s_k|J_\delta(\zeta')^{-1/2}\right) \\
&\quad + \epsilon c J_\delta(\zeta')\left(J_\delta(\zeta')^{-2}|\epsilon|^2 + \sum_{k=2}^{n-1}|s_k|^2 J_\delta(\zeta')^{-1} + \tau^{-2}|s_n|^2\right) \geq 0
\end{aligned}$$

provided that $J_\delta(\zeta')$ is sufficiently small (or equally if $|\zeta'|$ is sufficiently small). This completes the proof. □

Now we choose $\epsilon_0 > 0$ so that

$$\sup\{\rho(\zeta) : \zeta \in R_{C\delta}(z') \text{ and } \rho^{\epsilon_0}_{z'}(\zeta \leq 0\} < b\delta,$$

where b is the small number as in Proposition 2.3. This $\epsilon_0 > 0$ can be chosen independently of z' and δ, and we set $\rho_{z'}(\zeta) = \rho^{\epsilon_0}_{z'}(\zeta)$. Then the function $g_\delta(\zeta)$ (as in Proposition 2.3) is well defined on the set $\{\zeta : \rho_{z'}(\zeta) < 0\} = \Omega_{z',\delta}$. For ζ' near 0, define a polydisc $P_a(\zeta')$ by

(2.8)

$$\begin{aligned}
P_a(\zeta') = \{\zeta \in \mathbb{C}^n : &|\zeta_1 - \zeta_1'| < aJ_\delta(\zeta'),\ |\zeta_n - \zeta_n'| < \tau(z', aJ_\delta(\zeta')), \\
&|\zeta_k - \zeta_k'| < (aJ_\delta(\zeta'))^{1/2},\ \ k = 2,\ldots,n-1\}.
\end{aligned}$$

PROPOSITION 2.5. *There exist constants $a > 0$ and $d_1 > 0$ (independent of z', ζ' and δ) so that if $\zeta' \in \Omega_{z'}$ and $|\zeta'| < d_1$, then $\rho_{z'}(\zeta) < 0$ for $\zeta \in P_a(\zeta')$.*

Proof. We may assume that $\zeta' \in b\Omega_{z'}$ (this will be the worst case). If a is sufficiently small (independent of z' and δ), then

$$J_\delta(\zeta) \approx J_\delta(\zeta'), \quad \zeta \in P_a(\zeta'). \tag{2.9}$$

¿From the property (i) of Proposition 2.2, and with (2.9), there exists a small constant $c > 0$, such that

$$H_{z',\delta}(\zeta) \leq -cJ_\delta(\zeta'), \quad \zeta \in P_a(\zeta'). \tag{2.10}$$

By a simple Taylor's theorem argument, one can show that

$$|\rho(\zeta)| \leq CaJ_\delta(\zeta'), \quad \zeta \in P_a(\zeta'). \tag{2.11}$$

Since $\rho'_z(\zeta) = \rho(\zeta) + \epsilon_0 H_{z',\delta}(\zeta)$, using (2.10) and (2.11) we have $\rho'_z(\zeta) < 0$ if a is chosen so that $a < c\epsilon_0/C$. This completes th proof. □

The existence of the following two-sided bumping family of pseudoconvex domains was shown by the author in [4].

THEOREM 2.6. *Let Ω be a smoothly bounded pseudoconvex domain and let $z_0 \in b\Omega$ be a point of finite type. Then there is a neighborhood V of z_0 and a family of smoothly bounded pseudoconvex domains $\{\Omega_t\}_{-1 \leq t \leq 1}$ satisfying the following propoerties;*

(i) $\Omega_0 = \Omega$,
(ii) $\Omega_{t_1} \subset \Omega_{t_2}$ *if* $t_1 < t_2$,
(iii) $\{\partial\Omega_t\}_{-1 \leq t \leq 1}$ *is a* C^∞*family o f real hypersurfaces in* $\mathbb{C}^n$ *and the points of* $\partial\Omega_t \cap V$ *are finite type,*
(iv) $D_t - D_{-t} \subset V$ *for all* t.

REMARK 2.2.

(1) Property (iii) means that $\partial\Omega_t \longrightarrow \partial\Omega_{t_0}$ in C^∞-topology as t goes to t_0.
(2) There is a neighborhood V of $z_0 \in b\Omega$ such that the types of the points of $V \cap b\Omega$ are bounded and Theorem 2.6 holds on that neighborhood [3].

(3) From the construction of $\Phi_{z'}$ and $\rho_z'(\zeta)$, we can choose $d_1 > 0$ and a neighborhood $U \subset\subset V$ of z_0 (independent of z') so that ρ_z' is defined in $\{\zeta : |\zeta| < d_1\}$ and satisfies all the properties in this section for all $z' \in b\Omega \cap U$.

Set $\Omega_{t,z'} = \{\zeta \in \mathbb{C}^n : \Phi_{z'}(\zeta) \in \Omega_t\}$, where $\{\Omega_t\}$ is the family of domains as in Theorem 2.6. Let us denote $J_\delta(\zeta) = J_\delta(\zeta, z')$ to clarify the dependence of z'. Set

$$\Omega_{z',\delta} = \{\zeta : |\zeta| < d_1 \quad \text{and} \quad \rho_z'(\zeta) < 0\} \quad \text{and}$$

$$b\Omega_{z',\delta} = \{\zeta : |\zeta| < d_1 \quad \text{and} \quad \rho_z'(\zeta) = 0\}.$$

The construction of ρ_z' in this section shows that if $\zeta \in \overline{\Omega}_{z'}$ and if $d_1/2 < |\zeta| < d_1$, then

$$d(\zeta, b\Omega_{z',\delta}) \gtrsim J_\delta(\zeta, z').$$

Since $A_m(z') \gtrsim 1$ for all $z' \in U$, it follows that $J_\delta(\zeta, z') \gtrsim 1$ when $d_1/2 < |\zeta| < d_1$. Therefore there is a constant $c_1 > 0$ so that

$$d(\zeta, b\Omega_{z',\delta}) \geq c_1,$$

for $\zeta \in U \cap b\Omega$ and $d_1/2 < |\zeta| < d_1$. Choose $t = t_0$ sufficiently small so that

$$d(\zeta, b\Omega_{t_0,z'}) < c_1/2 \quad \text{if} \quad d_1/2 < |\zeta| < d_1.$$

Now define a domain $\tilde{\Omega}_{z',\delta}$ by

$$\tilde{\Omega}_{z',\delta} = \{\zeta \in \Omega_{t_0,z'} : |\zeta| \geq d_1\} \cup \{\Omega_{t_0,z'} \cap \Omega_{z',\delta}\}.$$

Since pseudoconvexity is a local condition, $\tilde{\Omega}_{z',\delta}$ is a pseudoconvex domain. By combining the properties of $\Omega_{z',\delta}$ and $\Omega_{t_0,z'}$, we obtain

PROPOSITION 2.7. *For all z' near z_0 and all δ, $0 < \delta < \delta_0$, the domai $\tilde{\Omega}_{z',\delta}$ has the following properties;*

(i) *$\tilde{\Omega}_{z',\delta}$ is a bounded pseudoconvex domain that contains $\Omega_{z'}$,*
(ii) *the function g_δ of Proposition 2.3 is defined on $\tilde{\Omega}_{z',\delta}$,*
(iii) *there is a constant $a > 0$ so that for all $\zeta' \in \Omega_{z'}$ with $|\zeta'| < d_1$, $P_a(\zeta') \subset \tilde{\Omega}_{z',\delta}$,*
(iv) *in the region $|\zeta| > d_1/2$, the boundaries $b\tilde{\Omega}_{z',\delta}$ are independent of δ and depend smoothly on z',*
(v) *in the region $\{\zeta : d_1/2 < |\zeta| < d_1\}$, the boundaries $b\tilde{\Omega}_{z',\delta}$ are of finite type, uniformly in z', and δ.*

3. Metric Estimates

For the lower bounds, it is enough to find lower bounds for $C_\Omega(z;X)$ because of (1.4). Assume that $r(z) = -b\delta/2$ and let z' be the projection of z onto $b\Omega$, and $\Phi_{z'}$ be its associated map. Here $b > 0$ is the number as in Proposition 2.3. Set $\zeta^\delta = (-b\delta/2, 0, \dots, 0) = (\zeta_1^\delta, \zeta_2^\delta \dots, \zeta_n^\delta)$. Then by (2.1), (2.2) and (2.3), there is a small constant $c \le b$ such that the polydisc

(3.1)

$$B = \{\zeta : |\zeta_1 + b\delta/2| < c\delta, |\zeta_n| < c\tau(z', \delta), |\zeta_k| < c\delta^{1/2},\ 2 \le k \le n-1\},$$

is contained in $\Omega_{z'}$ and hence the properties (iv) and (v) of Proposition 2.3 hold on B. Let $Y = (\Phi_{z'}^{-1})_* X = b_1 L_1' + \dots + b_n L_n'$ be a vector at ζ_δ, where $L_i' = (\Phi_{z'}^{-1})_{*_i}$ for $i = 1, 2, \dots, n$. From the coordinate changes as in Proposition 2.2 in [6], one has

$$\begin{aligned} L_1' &= \frac{\partial}{\partial \zeta_1}, \\ L_k' &= \sum_{j=2}^{n-1} \overline{P}_{kj} \lambda_j^{-1/2} \frac{\partial}{\partial \zeta_j} - \left(\frac{\partial \rho}{\partial \zeta_1}\right)^{-1} \\ &\sum_{j=2}^{n-1} \overline{P}_{kj} \lambda_j^{-1/2} \frac{\partial \rho}{\partial \zeta_j} \frac{\partial}{\partial \zeta_1},\ 2 \le k \le n-1, \\ L_n' &= \frac{\partial}{\partial \zeta_n} + b(\zeta) \frac{\partial}{\partial \zeta_1}, \end{aligned} \tag{3.2}$$

where $b(\zeta) = -\left(\frac{\partial \rho}{\partial \rho_1}\right)^{-1} \left(\frac{\partial \rho}{\partial \zeta_n}\right)$ and $P = (P_{kj})$ is a unitary matrix, and λ_j's are positive eigenvalues of $\partial\overline{\partial} r(z')$, $j = 2, \dots, n-1$. We may assume that $\lambda_j \ge c > 0$ on U for $j = 2, \dots, n-1$. Set $\tau_1 = \delta$, $\tau_n = \tau(z', \delta)$, $\tau_k = \delta^{1/2}$, $k = 2, \dots, n-1$. Let k_0 be the minimum number such that

$$|b_{k_0}| \tau_{k_0}^{-1} = \max\{|b_k| \tau_k^{-1} : k = 1, 2, \dots, n\}. \tag{3.3}$$

Set $v(\zeta) = \delta^{-1}(\zeta_1 + b\delta/2)$ if $k_0 = 1$, $v(\zeta) = \tau^{-1}\zeta_n$ if $k_0 = n$, and set $v(\zeta) = \delta^{-1/2} \sum_{j=2}^{n-1} P_{k_0 j} \lambda_j^{1/2} \zeta_j$ for $2 \le k_0 \le n-1$. Since we may assume that $c \leq 1$ and $\lambda_j \leq 1$, we have the inequality $\sup_B|v| \leq 1$. From the expansion in (2.1), one can see that $\frac{\partial \rho}{\partial \zeta_j}(\zeta^\delta) = 0$, $j = 2, \ldots, n$, and hence from (3.2),

$$|Yv(\zeta^\delta)| = \max\{|b|\tau_k^{-1} : k = 1, 2, \ldots, n\}, \tag{3.4}$$

provided that δ is sufficiently small. Set $\phi(\zeta) = g_\delta(\zeta) + |\zeta|^2$ and set $\lambda(\zeta) = \chi(\phi(\zeta))$, where $\chi(t)$ is a smooth convex increasing function with $\chi'(t) \geq 1$ and g_δ is the function as in Proposition 2.3. Using the standard $\overline{\partial}$-estimates on $\tilde{\Omega}_{z',\delta}$ with weight $e^{-\lambda(\zeta)}$, and from the estimate $\sum_{i,j=1}^n \frac{\partial^2 \phi}{\partial \zeta_i \partial \overline{\zeta}_j}(\zeta) t_i \overline{t}_j \gtrsim \sum_{j=1}^n |t_i|^2 \tau_i^{-2}$ for $\zeta \in B$, one has for any $g = \sum_{i=1}^n g_i \, d\overline{\zeta}_i \in D(T^*) \cap D(S)$ with $Sg = 0$,

$$\int_{\tilde{\Omega}_{z',\delta} - B} |g|^2 e^{-\lambda} dV + \int_B \sum_{i=1}^n \tau_i^{-2} |g_i|^2 e^{-\lambda} dV \lesssim \|T^* g\|_\lambda^2, \tag{3.5}$$

where T^* and S are densely defined operators induced from $\overline{\partial}^*$ and $\overline{\partial}$. Suppose $f = \sum_{i=1}^n f_i d\overline{\zeta}_i$ satisfies $Sf = 0$. Then from standard theory of $\overline{\partial}$ and (3.5), there is $u \in L^2_{0,0}(\tilde{\Omega}_{z',\delta}, \lambda)$ (=weighted L^2-space in $\tilde{\Omega}_{z',\delta}$) such that $\overline{\partial} u = f$ in weak sense) and,

$$\|u\|_\lambda^2 \lesssim \int_{\tilde{\Omega}_{z',\delta} - B} |f|^2 e^{-\lambda} + \int_B \sum_{i=1}^n \tau_i^2 |f_i|^2 e^{-\lambda} dV. \tag{3.6}$$

For $c \geq d > 0$, set $B_d = \{\zeta : |\zeta_i - \zeta_i^\delta| < d\tau_i, i = 1, 2, \ldots, n\}$. Since $\sum_{i,j=1}^n \frac{\partial^2 \phi}{\partial \zeta_i \partial \overline{\zeta}_j} t_i \overline{t}_j \gtrsim \sum_{i=1}^n \tau_i^{-2} |t_i|^2$ on B, there is a small constant $d > 0$ (independent of $\tau_1, \ldots, \tau_n$) so that

$$\phi(\zeta) \geq Re\ h(\zeta) + d \sum_{i=1}^n \tau_i^{-2} |\zeta_i - \zeta_i^\delta|^2, \quad \zeta \in B_d, \tag{3.7}$$

where

$$h(\zeta) = 2 \sum_{i=1}^n \frac{\partial \phi}{\partial \zeta_i}(\zeta^\delta)(\zeta_i - \zeta_i^\delta) \sum_{i,j=1}^n \frac{\partial^2 \phi}{\partial \zeta_i \partial \zeta_j}(\zeta^\delta)(\zeta_i - \zeta_i^\delta)(\zeta_j - \zeta_j^\delta).$$

Let $\psi \in C_0^\infty(U)$, where U is the unit polydisc in $\mathbb{C}^n$ such that $\psi(\zeta) = 1$ if $|\zeta_i| \leq 1/2$, $i = 1, 2, \ldots, n$. From (3.7), we conclude that if

$$\psi_d(\zeta) = \psi\left(\frac{\zeta_1 + b\delta/2}{d\tau_1}, \frac{\zeta_2}{d\tau_2}, \ldots, \frac{\zeta_n}{d\tau_n}\right),$$

and if $a = nd^3/8$, then

$$Re\ h(\zeta) \leq -a, \text{ for } \zeta \in \{\zeta : \phi(\zeta) \leq a\} \cap \operatorname{supp}\ \overline{\partial}\psi_d. \tag{3.8}$$

Let χ be a smooth convex increasing function that satisfies $\chi(t) = 0$ for $t \leq a/2$ and $\chi''(t) > 0$ for $t > a/2$. Now define

$$\lambda_s(\zeta) = \phi(\zeta) + s^2\chi(\phi(\zeta))$$

and set

$$\alpha_s = \overline{\partial}(\psi_d v e^{sh}) = v e^{sh}\overline{\partial}\psi_d = \sum_{i=1}^n \alpha_{s,i} d\overline{\zeta}_i.$$

Since $|\frac{\partial\psi_d}{\partial\zeta_i}| \lesssim \tau_i^{-1}$, it follows that $\alpha_s = \sum \alpha_{s,i} d\overline{\zeta}$ satisfies

$$\int_B \sum_{i=1}^n \tau_i^2 |\alpha_{s,i}|^2 e^{-\lambda_s} dV \lesssim \int_{\operatorname{supp}\ \overline{\partial}\psi_d} e^{2sReh - \phi - s^2\chi(\phi)} dV. \tag{3.9}$$

Suppose $\phi(\zeta) \geq 0$. Then $\chi(\phi(\zeta)) \geq \chi(a) > 0$, so the s^2-term is predominant. If $\phi(\zeta) \leq a$ and $\zeta \in \operatorname{supp}\ \overline{\partial}\psi_d$, then $Re\ h(\zeta) \leq -a$ by (3.8). So the integrand in the integral on the right-hand side of (3.9) approaches to zero uniformly as s converges to infinity. Hence from (3.6) and (3.9), we conclude that for any $\epsilon_0 > 0$, there exists $s_0 > 0$ (independent of $\tau_1, \ldots, \tau_n$) and a function u_{s_0} so that $\overline{\partial}u_{s_0} = \alpha_{s_0}$ and

(3.10)

$$\int_{\tilde{\Omega}_{z',\delta}} |u_{s_0}|^2 e^{-\lambda_{s_0}} dV \lesssim \int_B \sum_{i=1}^n \tau_i^2 |\alpha_{s_0,i}|^2 e^{-\lambda_{s_0}} dV$$
$$\lesssim \int_{\operatorname{supp}\ \overline{\partial}\psi_d} \epsilon_0 dV \lesssim \epsilon_0 \prod_{i=1}^n \tau_i^2.$$

From the property (v) of Proposition 2.3, there is $e > 0$ (independent of z' and δ) so that $\phi(\zeta) < a/2$ for all $z \in B_e = \{\zeta : |\zeta_i - \zeta_i^\delta| < e\tau_i,\ i = 1, 2, \dots, n\}$. Therefore $\lambda_s(\zeta)$ is independent of s for $\zeta \in B_e$ and hence u_{s_0} is holomorphic on B_e and satisfy

$$\left|\frac{\partial u_{s_0}}{\partial \zeta_k}(\zeta^\delta)\right|^2 \lesssim \tau_k^{-2}\left(\prod_{j=1}^n \tau_j^{-2}\right)\int_{B_e}|u_{s_0}|e^{-\lambda_{s_0}}dV$$

$$\lesssim \tau_k^{-2}\left(\prod_{j=1}^n \tau_j^{-2}\right)\left(\epsilon_0\prod_{j=1}^n \tau_j^2\right) = \epsilon_0\tau_k^{-2},$$

for $k = 1, 2, \dots, n$. Therefore it follows from (3.2) that

$$|Xu_{s_0}(\zeta^\delta)| \lesssim \sqrt{\epsilon_0}\sum_{k=1}^n |b_k|\tau_k^{-1} \le n\sqrt{\epsilon_0}\max\{|b_k|\tau_k^{-1} :\ k = 1, 2, \dots, n\}.$$

Set $f = v\psi_d e^{s_0 h} - u_{s_0}$. Then f is holomorphic and from (3.4), it follows that

$$|Yf(\zeta^\delta)| \gtrsim \max\{|b_k|\tau_k^{-1} : k = 1, 2, \dots, n\}, \tag{3.11}$$

provided that ϵ_0 is sufficiently small.

Let us assume, for a moment, that $\sup_{\overline{\Omega}_{z'}}|f| \le C$, where C is independent of z' and δ. Then (3.11) and the definition of Caratheodory metric shows that

$$C_{\Omega_{z'}}(Y; \zeta^\delta) \ge C_{\tilde{\Omega}_{z',\delta}}(Y; \zeta^\delta) \gtrsim \max\{|b_k|\tau_k^{-1} : k = 1, 2, \dots, n\}. \tag{3.12}$$

On the other hand, the polydisc B about ζ^δ lies in $\Omega_{z'}$. So one can easily obtain that

$$C_{\Omega_{z'}}(\zeta^\delta; Y) \le C_B(\zeta^\delta; Y) = \max\{|b_k|\tau_k^{-1} : k = 1, 2, \dots, n\}. \tag{3.13}$$

From (1.1), (1.2), (2.3) and (2.4), we have

$$\max\{|b_k|\tau_k^{-1} : k = 1, 2, \dots, n\} \approx M_m(z; X)$$

and hence from the invariant property of Caratheodory metric, and with (3.12), (3.13), one has

$$C_\Omega(z;X) = C_{\Omega_{z'}}(\zeta^\delta;Y) \approx M_m(z;X). \tag{3.14}$$

To show that $\sup_{\overline{\Omega}_{z'}}|f| \le C$, we use the fact that f is holomorphic in a larger domain $\tilde{\Omega}_{z',\delta}$. Assume $\zeta \in \overline{\Omega}_{z'}$ and $|\zeta| < d_1$. Then from Proposition 2.7, one can see that $P_a \subset \tilde{\Omega}_{z',\delta}$. Since $|v\psi_d e^{s_0 h}| \lesssim 1$ and from the estimate (3.10), it follows that $\int_{P_a(\zeta)} |f|^2 dV \lesssim \prod_{j=1}^n \tau_j^2$, and hence

$$|f(\zeta)| \lesssim (Vol(P_a(\zeta))^{-1} \int_{P_a(\zeta)} |f|^2 dV \lesssim 1,$$

because $Vol(P_a(\zeta)) \gtrsim \prod_{j=1}^n \tau_j^2$. When $|\zeta| \ge d_1$, we use the Kohn's global regularity theory and some cut-off functions as Catlin did in [2]. Therefore we proved that $\sup_{\overline{\Omega}_{z'}}|f| \lesssim 1$ and hence (3.14) has been proved.

To obtain an upper ound for the Bergman metric, we note that $\Omega_{z'}$ contains the polydisc B about ζ^δ. Thus by elementary estimates, one has for any $f \in L^2(\Omega_{z'}) \cap A(\Omega_{z'})$,

$$\left|\frac{\partial f}{\partial \zeta_k}(\zeta^\delta)\right| \lesssim \tau_k^{-1} \prod_{j=1}^n \tau_j^{-1} \|f\|_{L^2(\Omega_{z'})},$$

for $k = 1, 2, \dots, n$. From (2.1) and (3.2), it follows that the coefficient $b(\zeta)$ of $\frac{\partial}{\partial \zeta_1}$ in L_n' satisfies $|b(\zeta^\delta)| \lesssim \delta$ and $|\frac{\partial \rho}{\partial \zeta_j}(\zeta^\delta)| \lesssim \delta^{1/2}$, for $j = 2, \dots, n-1$. Therefore, if $Y = \sum_{k=1}^n b'_{k_k}$ is a vector atζ^δ, then

$$b_{\Omega_{z'}}(\zeta^\delta;Y) \lesssim \left(\sum_{k=1}^n |b_k|\tau_k^{-1}\right) \prod_{j=1}^n \tau_j^{-1}. \tag{3.15}$$

In [6], the author showed that

$$K_{\Omega_{z'}}(\zeta^\delta, \overline{\zeta}^\delta) \approx \prod_{j=1}^n \tau_j^{-2}. \tag{3.16}$$

Combining (3.15), (3.16) and from the definition of $B_\Omega(z;X)$, it follows that

$$B_\Omega(z;X) = B_{\Omega_{z'}}(\zeta^\delta;Y) \lesssim \sum_{k=1}^{n} |b_k|\tau_k^{-1},$$

and hence one has

$$C_\Omega(z;X) \approx B_\Omega(z;Y) \approx M_m(z;X). \tag{3.17}$$

To show $K_\Omega(z;X) \approx M_m(z;X)$, we set

$$a_k = -\left(\frac{\partial\rho}{\partial\zeta_1}(\zeta^\delta)\right)^{-1} \sum_{j=2}^{n-1} \overline{P}_{kj}\lambda_j^{-1/2}\frac{\partial\rho}{\partial\rho_j}(\zeta^\delta), \quad k = 2,\dots,n-1,$$

and set

$$b_0 = -\left(\frac{\partial\rho}{\partial\zeta_1}(\zeta^\delta)\right)^{-1}\left(\frac{\partial\rho}{\partial\zeta_n}(\zeta^\delta)\right).$$

Therefore we hae $|a_k|$, $|b_0| \lesssim \delta$ on B. Set

$$R = \min\{d_2 c\tau_k|b_k|^{-1} : k = 1,2,\dots,n\}.$$

Then

$$f(t) = (-b\delta/2 + (b_1 + \sum_{k=2}^{n-1} a_k b_k + b_n d_0)t, \lambda_2^{-1/2}\sum_{=2}^{n-1} b_k\overline{P}_{k2}t,$$
$$\dots, \lambda_{n-1}^{-1/2}\sum_{k=2}^{n-1} b_k\overline{P}_{k,n-1}t, b_n t)$$

defines a map $f : D_R \longrightarrow B$ with $f_*(\frac{\partial}{\partial t}|_0) = X$ provided that d_2 is sufficiently small. Hence

$$\begin{aligned} K_{\Omega_{z'}}(\zeta^\delta;Y) &\le K_B(\zeta^\delta;Y) \le R^{-1} \le \max\{|b_k|(cd_2\tau_k)^{-1} :\ 1 \le k \le n\} \\ &\lesssim \max\{|b_k|\tau_k^{-1} :\ k = 1,2,\dots,n\} \\ &\lesssim \sum_{k=1}^{n} |b_k|\tau_k^{-1} \lesssim C_{\Omega_{z'}}(\zeta^\delta;Y). \end{aligned}$$

Again from the invariant property of $K_\Omega(z;X)$ and (1.4), it follows that

$$K_\Omega(z;X) = K_{\Omega_{z'}}(\zeta^\delta;Y) \approx C_\Omega(z;X) \tag{3.18}$$

If one combines (3.17) and (3.18), one will get

$$C_\Omega(z;X) \approx B_\Omega(z;X) \approx K_\Omega(z;X) \approx M_m(z;X),$$

and this proves our main theorem. □

REMARK 2.3. It seems that Kohn's ideal type and D'Angelo's finite 1-type are same in our case. This will be discussed in a forthcoming article.

References

1. Bedford, E. and Fornaess, J. E., *Biholomorphic maps of weakly pseudoconvex domains*, Duke Math. J. **45** (1978), 711-719.
2. Catlin, D. W., *Estimates of invariant metrics on pseudoconex domains of dimension t wo*, Math. Z. **200** (1989), 429-466.
3. Cho, S., *On the extension of complex structures on weakly pseudoconvex compact complex manifolds with boundary*, Dissertation, Purdue University (1991).
4. Cho, S., *Extension of complex structures on weakly pseudoconvex compact complex manifolds with boundary*, Math. Z. **211** (1992), 105-120.
5. Cho, S, *A lower bound on the Kobayashi metric near a point of finite type in* $\mathbb{C}^n$, The J. of Geom. Analysis **2, No 4** (1992), 317-325.
6. Cho, S., *Boundary behavior of the Bergman kernel function on some pseudoconvex domains in* $\mathbb{C}n$, Transac. of A.M.S. (to appear).
7. D'Angelo, J., *Real hypersurfaces, order of contact, and applications*, Ann. of Math. **115** (1982), 615-637.
8. Diederich, K. and Fornaess, J., *Proper holomorphic maps onto pseudoconvex domains with real-analytic boundary*, Ann. of Math. **110** (1979), 575-592.
9. Diederich, K., Fornaess, J., and Herbort, G., *Boundary behavior of the Bergman metric*, Proc. Symposia in Pure Math. **41** (1982), 59-67.
10. Hahn, K. T., *Inequalities between the Bergman metric and the Caratheodory differential metric*, Proc. Amer. Math. Soc. **68** (1978), 193-194.
11. Herbort, G., *On the invariant differential metrics near pseudoconvex boundary points where the Levi form has corank one*, Nagoya Math. J **130** (1993), 25-54.
12. Range, R. M., *The Caratheodory metric and holomorphic maps on a class of weakly pseudoconvex domains*, Pac. J. Math. **78** (1978), 173-189.

Department of Math. Education
Pusan National University, Pusan, 609-735, Korea
e-mail: cho@hyowon.cc.pusan.ac.kr