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WEAK TYPE INEQUALITY FOR CERTAIN MAXIMAL FUNCTIONS ON SPACES OF HOMOGENEOUS TYPE

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1. Introduction

Classical Fatou's theorem asserts that the non-tangential limit of the Poisson integral of an integrals function on $[-\pi, \pi]$ exists a.e. In [NS] Nagel and Stein generalized this theorem by showing that there are certain approach region $\Omega \subset \mathbf{R}_+^{n+1}$, which are not contained in any non-tangential region, but for which the limit

$$\lim_{\Omega \ni (x,r) \to (x_0,0)} u(x,r)$$

exists a.e. on $\mathbf{R}^n$. For the complex case Korányi showed a similar result in [K].

In [Su] Sueiro studied a certain maximal operator M_Ω which generalizes the classical Hardy-Littlewood maximal operator to study Poisson-Szego integral on the unit ball in $\mathbf{C}^n$ or on the Siegel half-space by showing that $\mathcal{M}_\Omega$ is of weak type (1,1). In this paper we are going to characterize condition to be $\mathcal{M}_\Omega$ of weak type (p,q), $p, q \geq 1$. In this direction see also [HMS, MW, R, W, We].

2. Preliminaries

Let (X, d, μ) be a space of homogeneous type, i.e., X is a topological space, μ is a positive Borel measure on X, and d is a pseudo-metric on X; more precisely, we assume that there are constants A and K so that for all $x, y, z \in X$ and all $\delta > 0$,

i) $d(x,y) \geq 0; d(x,y) = 0$ if and only if $x = y$;

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ii) $d(x,y) = d(y,x)$;
iii) $d(x,z) \le K[d(x,y) + d(y,z)]$;
iv) $\mu(B(x,2\delta)) \le A\mu(B(x,\delta)), \quad \delta > 0$,

where $B(x,\delta) = \{y \in X : d(x,y) < \delta\}$ form a basis for the topology of X. Property iv) is referred to the doubling property of μ. For details, see [CW1].

Suppose that there is a given set $\Omega_{x_0} \subset X \times (0,\infty) \equiv X^+$ for each $x_0 \in X$. Let Ω be the family $\{\Omega_{x_0} : x_0 \in X\}$. We define a maximal function associated with Ω as follows. For $f \in L^1_{loc}(X, d\mu)$ and $x_0 \in X$ set

$$\mathcal{M}_\Omega f(x_0) = \sup_{(x,r)\in\Omega_{x_0}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| d\mu,$$

where $|B(x,\delta)| = \mu(B(x,\delta))$ for simplicity.

For $(x,r) \in X^+$, and $\alpha > 0$, define

$$S_\alpha(x,r) = \{x_0 \in X : \Omega_{x_0}(r) \cap B(x,\alpha r) \ne \phi\},$$

where

$$\Omega_{x_0}(r) = \{x \in X : (x,r) \in \Omega_{x_0}\}$$

is the cross section of Ω_{x_0} of height $r > 0$.

Let (u,w) be a pair of non-negative measurable functions on X. Let T be an operator. If there exists a constant C so that

$$u(\{|Tf| > \lambda\}) \le C\{\frac{\|f\|_{p,w}}{\lambda}\}^q, \quad 1 \le p \le \infty, \quad 1 \le q < \infty,$$

then T is said to be of weak type (p,q) with respect to weight (u,w). For $q = \infty$, if $\|Tf\|_{q,u} \le C\|f\|_{p,w}$, then T is also said to be weak type $(p,q) = (p,\infty)$ with respect to weight (u,w). Denote

$$u(A) = \int_A u d\mu \quad \text{and} \quad \|f\|_{p,w} = (\int |f|^p w d\mu)^{\frac{1}{p}}$$

as usual.

A pair of weights (u,w) is of the class $A_{p,q}(\Omega)$ if

$$\frac{u(S_\alpha(x,r))}{|B(x,r)|^q} \Big(\int_{B(x,b(\alpha)r)} w^{-\frac{1}{p-1}} d\mu \Big)^{q(1-\frac{1}{p})} \le C(\alpha,K),$$

if $1 < p < \infty, 1 \leq q < \infty$ and

$$\frac{u(S_\alpha(x,r))}{w(B(x,b(\alpha)r))^q} \leq C(\alpha, K), \quad \text{if} \quad p = 1, 1 \leq q < \infty,$$

for some constants $C(\alpha, K)$ and $b(\alpha) = b(\alpha, K)$.

Note that the constants need not be same at each occurance

3. Weak type inequality of $\mathcal{M}_\Omega$

The following lemma is given in [CW2]. See also [Su] for slightly improved one.

LEMMA 1. *Let E be a bounded subset of X. Suppose that $r(x)$ is a positive number for each $x \in E$. Then there is a sequence of disjoint balls $\{B(x_i, r(x_i))\}$, $x_i \in E$, such that*

$$E \subset \cup_i B(x_i, 4Kr_i), \quad r_i = r(x_i),$$

where K is the constant in the triangle inequality (iii). Furthermore, every $x \in E$ is contained in some ball $B(x_i, 4Kr_i)$ satisfying $r(x) \leq 2r_i$.

THEOREM 1. *Suppose $\mathcal{M}_\Omega$ is of weak type (p,q) with respect to a weight (u,w), $1 < p, q < \infty$, with respect to a pair of weights (u,w). Then $(u,w) \in A_{p,q}(\Omega)$.*

Proof. Assume that f is nonnegative without loss of generality. Let $\mathcal{M}_\Omega$ is of weak type (p,q). Let $x_0 \in S_\alpha(x,r)$. Then there exists $y \in X$ such that

$$y \in \Omega_{x_0}(r) \cap B(x, \alpha r).$$

If $d(y,z) < r$, then

$$\begin{aligned} d(z,x) &\leq K(d(z,y) + d(y,x)) \\ &\leq K(\alpha r + r) \\ &= rK(\alpha + 1), \end{aligned}$$

and thus

$$B(y,r) \subset B(x, K(\alpha + 1)r). \tag{1}$$

In the same fashion,

$$B(x,\delta) \subset B(y, K(\delta+\alpha r))$$

for all $\delta > 0$. If we choose δ so that $K(\delta + \alpha r) = r$, then

$$B(x, b(\alpha)r) \subset B(y,r)$$

for some $b(\alpha)$. In fact, $b(\alpha) = \frac{1}{K} - \alpha$. Now let χ_A be the characteristic function of a set A. Then from (1) it follows that

$$\begin{aligned} \mathcal{M}_\Omega(f\chi_{B(x,K(\alpha+1)r)})(x_0) &\geq \frac{1}{|B(y,r)|}\int_{B(y,r)} f\chi_{B(x,K(\alpha+1)r)}d\mu \\ &\geq \frac{1}{|B(y,r)|}\int_{B(y,r)} f d\mu. \end{aligned}$$

Put

$$f_{B(y,r)} = \frac{1}{|B(y,r)|}\int_{B(y,r)} f d\mu$$

for simplicity. If

$$0 < t < f_{B(y,r)},$$

then

$$S_\alpha(x,r) \subset \{\mathcal{M}_\Omega(f\chi_{B(x,K(\alpha+1)r)}) > t\}$$

and so

$$\begin{aligned} u(S_\alpha(x,r) &\leq u(\{M_\Omega(f\chi_{B(x,K(\alpha+1)r)}) > t\}) \\ &\leq \frac{C}{t^q}\Big(\int_{B(x,K(\alpha+1)r)} f^p w d\mu\Big)^{\frac{q}{p}}. \end{aligned}$$

Thus

$$t^q u(S_\alpha(x,r)) \leq C\Big(\int_{B(x,K(\alpha+1)r)} f^p w d\mu\Big)^{\frac{q}{p}}$$

and therefore

$$f^q_{B(y,r)} u(S_\alpha(x,r)) \leq C\Big(\int_{B(x,K(\alpha+1)r)} f^p w d\mu\Big)^{\frac{q}{p}},$$

or

$$(2)\qquad \frac{u(S_\alpha(x,r))}{|B(y,r)|^q}\Big(\int_{B(y,r)} f\,d\mu\Big)^q \le C\Big(\int_{B(x,K(\alpha+1)r)} f^p w\,d\mu\Big)^{\frac{q}{p}}.$$

Since $B(y,r) \subset B(x,K(\alpha+1)r)$ by (1), if we replace f by $f\chi_{B(y,r)}$ in (2), then

$$(3)\qquad \frac{u(S_\alpha(x,r))}{|B(y,r)|^q}\Big(\int_{B(y,r)} f\,d\mu\Big)^q \le C\Big(\int_{B(y,r)} f^p w\,d\mu\Big)^{\frac{q}{p}}.$$

Replace again f by $f\chi_{B(x,b(\alpha)r)}$ in (3) to obtain

$$(4)\qquad \frac{u(S_\alpha(x,r))}{|B(y,r)|^q}\Big(\int_{B(x,b(\alpha)r)} f\,d\mu\Big)^q \le C\Big(\int_{B(x,b(\alpha)r)} f^p w\,d\mu\Big)^{\frac{q}{p}}.$$

Since

$$\begin{aligned}|B(y,r)| &\le |B(x,K(\alpha+1)r)|\\ &\le C|B(x,r)|\end{aligned}$$

by the doubling property, we have

$$(5)\qquad \frac{u(S_\alpha(x,r))}{|B(x,r)|^q}\Big(\int_{B(x,b(\alpha)r)} f\,d\mu\Big)^q \le C\Big(\int_{B(x,b(\alpha)r)} f^p w\,d\mu\Big)^{\frac{q}{p}}.$$

Suppose $p > 1$. Replacing f by $w^{\frac{1}{1-p}}$ so that $f = f^p w$, we obtain

$$\frac{u(S_\alpha(x,r))}{|B(x,r)|^q}\Big(\int_{B(x,b(\alpha)r)} w^{-\frac{1}{p-1}}\,d\mu\Big)^{q(1-\frac{1}{p})} \le C.$$

Suppose $p = 1$. If $q \ge 1$, then set $f \equiv 1$ to obtain

$$\frac{u(S_\alpha(x,r))}{w(B(x,b(\alpha)r))^q} \le C,$$

and this completes the proof. ||||

THEOREM 2. *Assume that* Ω *satisfies the following conditions i) and ii):*

i) *If* $x_0 \in X$, $(x,r) \in \Omega_{x_0}$ *and* $s \geq r$, *then* $(x,s) \in \Omega_{x_0}$

ii) *A weight* $(u,w) \in A_{p,q}(\Omega), p,q > 1$.

Then $\mathcal{M}_\Omega$ *is of weak type* (p,q).

Proof. The proof is motivated in the argument in [Su]. Assume $f \geq 0$ without loss of generality. Suppose $(u,w) \in A_{p,q}$.

Let

$$E_\lambda = \Big\{x \in X : \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f d\mu > \lambda\Big\}$$

for any $\lambda > 0$. For each $x \in E_\lambda$ let

$$r(x) = \sup\Big\{r > 0 : \frac{1}{|B(x,r)|} \int_{B(x,r)} f d\mu > \lambda\Big\}$$

Then following the argument in the proof of theorem 1.5 [Su], we have

$$\{M_\Omega f > \lambda\} \subset \cup_i S_\alpha\Big(x_i, \frac{4K}{\alpha} r_i\Big). \tag{1}$$

Thus

$$\begin{aligned} &u(\{\mathcal{M}_\Omega f > \lambda\}) \\ &\leq \sum_i u\Big(S_\alpha(x_i, \frac{4K}{\alpha} r_i)\Big) \\ &\leq C \sum_i \Big|B(x_i, \frac{4K}{\alpha} r_i)\Big|^q \Big(\int_{B(x_i, b(\alpha)\frac{4K}{\alpha} r_i)} w^{-\frac{1}{p-1}} d\mu\Big)^{-q(1-\frac{1}{p})}. \end{aligned} \tag{2}$$

Since

$$b(\alpha)\frac{4K}{\alpha} = \Big(\frac{1}{K} - \alpha\Big)\frac{4K}{\alpha} = 4(\frac{1}{\alpha} - K) \geq 1,$$

we have

$$B\Big(x_i, b(\alpha)\frac{4K}{\alpha} r_i\Big) \supset B(x_i, r_i). \tag{3}$$

Then it follows from (3) and Hölder inequality that

(4)
$$
\begin{aligned}
&u(\{\mathcal{M}_\Omega f > \lambda\}) \\
&\le C \sum_i |B(x_i, r_i)|^q \Big(\int_{B(x_i,r_i)} w^{-\frac{1}{p-1}} d\mu \Big)^{-q(1-\frac{1}{p})} \\
&\le \frac{C}{\lambda^q} \sum_i \Big(\int_{B(x_i,r_i)} f^p w d\mu \Big)^{\frac{q}{p}} \Big(\int_{B(x_i,r_i)} w^{-\frac{1}{p-1}} d\mu \Big)^{q(\frac{p-1}{p})-q(\frac{p-1}{p})} \\
&\le \frac{C}{\lambda^q} \Big(\int_X f^p w d\mu \Big)^{\frac{q}{p}}. \\
&= \frac{C}{\lambda^q} \|f\|^q_{L^p(wd\mu)}.
\end{aligned}
$$

Note that the last inequality in (4) is used by the fact that $\{B(x_i, r_i)\}$ is a disjoint family. This proves that $\mathcal{M}_\Omega$ is of weak type (p, q) with respect to the weight (u, w).

For the unbounded case a similar argument is valid and this completes the proof. ||||

For $x_0 \in X$, set

$$\hat{\Omega}_{x_0} = \{(x, r) \in X^+ : (x, s) \in \Omega_{x_0} \quad \text{for some} \quad s \le r\}$$

and set

$$\hat{S}_\alpha(x, r) = \{x_0 \in X : \hat{\Omega}_{x_0}(r) \cap B(x, \alpha r) \ne \phi\}.$$

In this context we define a maximal function $\mathcal{M}_{\hat{\Omega}} f$ by

$$\mathcal{M}_{\hat{\Omega}} f(x_0) = \sup_{(x,r)\in\hat{\Omega}_{x_0}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f| d\mu.$$

Also we can define $\hat{A}_{p,q}$ condition as follows. A weight (u, w) is of the class $A_{p,q}(\hat{\Omega})$ if

$$\frac{u(\hat{S}_\alpha(x,r))}{|B(x,r)|^q} \Big(\int_{B(x,b(\alpha)r)} w^{-\frac{1}{p-1}} d\mu \Big)^{q(1-\frac{1}{p})} \le C(\alpha, K),$$

if $1 < p < \infty$, $1 \le q < \infty$, and

$$\frac{u(\hat{S}_\alpha(x,r))}{(w(B(x,b(\alpha)r)))^q} \le C(\alpha,K),$$

if $p = 1, 1 \le q < \infty$ for some constant $C(\alpha, K)$ and constant $b(\alpha) = b(\alpha, K)$.

THEOREM 3. *If $\mathcal{M}_\Omega$ is of weak type (p,q) with respect to weight (u,w) then $(u,w) \in A_{p,q}(\hat{\Omega})$.*

Proof. As in the proof of theorem 1, if $x_0 \in \hat{S}_\alpha(x,r)$, then $B(y,r) \subset B(x, K(\alpha+1)r)$ for some $(y,r) \in \hat{\Omega}_{x_0}$. But then $(y,s) \in \Omega_{x_0}$ for $s \le r$. Hence $B(y,s) \subset B(x,K(\alpha+1)r)$. Also $B(x,b(\alpha)r) \subset B(y,r)$ for some $b(\alpha)$. The remaining part of the proof follows those arguments in the proof of theorem 1. ||||

THEOREM 4. *If $(u,w) \in A_{p,q}(\hat{\Omega})$, then $\mathcal{M}_\Omega$ is of weak type (p,q) with respect to (u,w).*

Proof. Suppose $(u,w) \in A_{p,q}(\hat{\Omega})$. Since $S_\alpha(x,r) \subset \hat{S}_\alpha(x,r), (u,w)$ satisfies the condition (ii) of theorem 2. Hence $\mathcal{M}_{\hat{\Omega}}$ is of weak type (p,q). Since $\mathcal{M}_\Omega f \le \mathcal{M}_{\hat{\Omega}} f$ for all f, $\mathcal{M}_\Omega$ is of weak type (p,q). This completes the proof. ||||

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