

A SHARP (H^1, L^1) CONTINUITY THEOREM FOR THE RUBIO DE FRANCIA MAXIMAL MULTIPLIER OPERATOR

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1. Introduction

Consider a function m defined in \mathbf{R}^n and the associated convolution operator T given by the Fourier transform $\hat{T} = m \hat{f}$. The well-known Hörmander-Marcinkiewicz multiplier theorem asserts that if m satisfies

$$(1-1) \quad \sup_{0 < R < \infty} \int_{R \leq |\xi| \leq 2R} |D^\alpha m(\xi)|^2 d\xi \leq B_\alpha R^{-2|\alpha|+n}$$

for all multi-indices α , $|\alpha| \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$, then T extends to a bounded operator from $L^p(\mathbf{R}^n)$ to itself when $1 < p < \infty$ and is of weak type $(1, 1)$ (See [3], [6] for details). A typical case occurs when m verifies a stronger condition

$$(1-2) \quad |D^\alpha m(x)| \leq C_\alpha |x|^{-|\alpha|}, \quad |\alpha| \leq l = \left\lfloor \frac{n}{2} \right\rfloor.$$

Let us turn our attention to the corresponding maximal theory. We shall be concerned about the operator described as

$$(1-3) \quad T^* f(x) = \sup_{t>0} |T_t f(x)|, \quad (T_t f) \hat{\ }(\xi) = m(t\xi) \hat{f}(\xi).$$

In this connection let us point out a theorem of Rubio de Francia [5] which states that if $m \in C^{l+1}(\mathbf{R}^n)$ and for all α , $|\alpha| \leq l + 1$,

$$(1-4) \quad |D^\alpha m(x)| \leq C_\alpha (1 + |x|)^{-a} \quad \text{with some fixed } a > 1/2,$$

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then T^* extends continuously to a mapping from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ provided $2n/(n + 2a - 1) < p < (2n - 2)/(n - 2a)$.

The purpose of this note is to set up the (H^1, L^1) -inequality for T^* under a mild assumption on m in analogy with Hörmander-Marcinkiewicz hypothesis (1-1). Specifically we shall prove

THEOREM 1. *Suppose that there exist positive numbers a, b, δ with $a + b > 1, \delta > 3/2$ such that*

- (1) $|m(\xi)| \leq C(1 + |\xi|)^{-a}, \quad |\nabla m(\xi)| \leq C(1 + |\xi|)^{-b},$
- (2) *for all multi-indices α of order $|\alpha| = l, l + 1,$*

$$\sup_{1 \leq R < \infty} \int_{R \leq |\xi| \leq 2R} |D^\alpha m(\xi)|^2 d\xi \leq A_\alpha R^{-2\delta}.$$

Then we have the a priori continuity inequality

$$\|T^* f\|_{L^1(\mathbb{R}^n)} \leq B \|f\|_{H^1(\mathbb{R}^n)}.$$

We remark here that the first hypothesis is required to ensure the L^2 boundedness (see [5], pp. 398). It is interesting to note that the second hypothesis requires decays on the derivatives of m of only highest orders. Our method reveals that it's possible to have a slightly weaker assumption, namely,

THEOREM 2. *The hypothesis (2) of Theorem 1 may be replaced by*

$$(2)' \quad \sup_{1 \leq R < \infty} \left(\int_{R \leq |\xi| \leq 2R} |D^\alpha m(\xi)|^{2-\epsilon} d\xi \right)^{1+\epsilon} \leq A_\alpha R^{-2\delta}$$

for an arbitrarily small $\epsilon > 0$ and $\alpha, |\alpha| = l, l + 1.$

By the standard interpolation and well-known duality argument between the Hardy space $H^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$, the above theorems implicate instantly that

$$\|T^* f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty$$

and moreover the pointwise convergence property

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^n} f(x - ty) K(y) dy = f(x) \quad \text{a.e.,}$$

where $\hat{K} = m, f \in L^p(\mathbb{R}^n), 1 < p < \infty.$ Throughout this paper, C will stand for a positive constant which may be different in each occurrence.

2. Preliminary Settings

In preparation for the proof, we begin with collecting more or less known results and techniques which are relevant to our purposes (refer [5], [6] and [7]). Choose a radial bump function $\psi \in C^\infty(\mathbb{R}^n)$ satisfying

$$\psi(\xi) = \begin{cases} 1, & |\xi| \leq 1 \\ 0, & |\xi| > 2 \end{cases}$$

Let us set $\phi(\xi) = \psi(\xi) - \psi(2\xi)$ so that $\text{supp}(\phi) \subset \{1/2 < |\xi| < 2\}$ and

$$\psi(\xi) + \sum_{j \geq 1} \phi(2^{-j}\xi) = 1 \quad \text{for all } \xi \neq 0.$$

It follows that we have a Littlewood-Paley decomposition for m ,

$$m(\xi) = \psi(\xi)m(\xi) + \sum_{j \geq 1} \phi(2^{-j}\xi)m(\xi) = \psi(\xi)m(\xi) + \sum_{j \geq 1} m_j(\xi)$$

and consequently we get the majorization $T^*f(x) \leq M^*f(x) + \sum_{j \geq 1} T_j^*f(x)$, where M^* , T_j^* denote the associated dyadic maximal operators.

It turns out that M^* behaves extremely nice (see [5], pp. 398) so we only concentrate on studying T_j^* , $j \geq 1$. A simple application of the fundamental theorem of calculus shows that

$$\begin{aligned} T_j^*f(x) &\leq \left(2S_j f(x) \tilde{S}_j f(x)\right)^{1/2} \\ &= 2^{j/2}(S_j f)(x) + 2^{-j/2}(\tilde{S}_j f)(x), \end{aligned}$$

where

$$\begin{aligned} S_j f(x) &= \left(\int_0^\infty |T_j^t f(x)|^2 \frac{dt}{t}\right)^{\frac{1}{2}}, \\ \tilde{S}_j f(x) &= \left(\int_0^\infty |\tilde{T}_j^t f(x)|^2 \frac{dt}{t}\right)^{\frac{1}{2}}, \end{aligned}$$

$$(T_j^t f)^\wedge(\xi) = m_j(t\xi)\hat{f}(\xi),$$

$$(T_j^t f)^\wedge(\xi) = \tilde{m}_j(t\xi)\hat{f}(\xi),$$

$$\tilde{m}_j(\xi) = \nabla m_j(\xi) \cdot \xi.$$

LEMMA 1. *The condition (1) of Theorem 1 implies that T^* is bounded in $L^2(\mathbb{R}^n)$.*

Proof. The proof follows from the easy observation

$$(2-1) \quad \|T_j^* f\|_2 \leq C \|m_j\|_\infty^{1/2} \|\tilde{m}_j\|_\infty^{1/2} \|f\|_2. \quad \square$$

Let us turn now to a brief description of the Hardy space $H^1(\mathbb{R}^n)$. This space is an extremely nice subspace of $L^1(\mathbb{R}^n)$, elements of which are stable under various singular and maximal operators. In particular,

$$H^1(\mathbb{R}^n) = \{ f \in L^1(\mathbb{R}^n) \mid R_j f \in L^1(\mathbb{R}^n), \quad j = 1, 2, \dots, n \},$$

where R_j denotes the Riesz transforms $(R_j f)(\xi) = c_n \frac{\xi_j}{|\xi|}$, $1 \leq j \leq n$. More importantly it admits the atomic decomposition for each element. A function a is called an $H^1(\mathbb{R}^n)$ -atom if there exists a cube Q such that

$$(1) \quad \text{supp}(a) \subset Q$$

$$(2) \quad \int_Q a(x) dx = 0, \quad \|a\|_{L^2(\mathbb{R}^n)} \leq |Q|^{-1/2}.$$

According to R. Latter [4], any $f \in H^1(\mathbb{R}^n)$ can be decomposed into $f = \sum \lambda_Q a_Q$ where a_Q is an atom and scalars λ_Q satisfy $\sum_Q |\lambda_Q| \leq C \|f\|_{H^1} \approx \|f\|_{L^1(\mathbb{R}^n)} + \sum_{j=1}^n \|R_j f\|_{L^1(\mathbb{R}^n)}$. We shall need the following simple fact

LEMMA 2. *Suppose that T is an L^2 -bounded sublinear operator and for each H^1 -atom a_Q supported in a cube Q ,*

$$\int_{{}^c\bar{Q}} |T a_Q(x)| dx \leq A,$$

where ${}^c\bar{Q}$ denotes the complement of \bar{Q} and \bar{Q} the 2-fold concentric dilate of Q . Then T maps $H^1(\mathbb{R}^n)$ continuously into $L^1(\mathbb{R}^n)$. More precisely, we have $\|T f\|_{L^1} \leq C (\|T\|_{2,2} + A) \|f\|_{H^1}$.

Proof. For $f = \sum \lambda_Q a_Q$ belonging to the Hardy space, we have

$$\begin{aligned} \|Tf\|_{L^1} &\leq \sum |\lambda_Q| \int |T a_Q(x)| dx \\ &= \sum |\lambda_Q| \left(\int_{\bar{Q}} |T a_Q(x)| dx + \int_{c\bar{Q}} |T a_Q(x)| dx \right). \end{aligned}$$

It follows from the Cauchy-Schwartz inequality that

$$\begin{aligned} \int_{\bar{Q}} |T a_Q(x)| dx &\leq |\bar{Q}|^{1/2} \|T a_Q\|_2 \\ &\leq \|T\|_{2,2} |\bar{Q}|^{1/2} \|a_Q\|_2 \\ &\leq \|T\|_{2,2} |\bar{Q}|^{1/2} |Q|^{-1/2} \leq C \|T\|_{2,2}. \end{aligned}$$

Therefore,

$$\|Tf\|_{L^1} \leq C (\|T\|_{2,2} + A) \sum |\lambda_Q| \leq C (\|T\|_{2,2} + A) \|f\|_{H^1}. \quad \square$$

3. Proof of The Main Theorem

Notice first that

$$(3-1) \quad \|T_j^* f\|_1 \leq 2^{j/2} \|S_j f\|_1 + 2^{-j/2} \|\tilde{S}_j f\|_1$$

and we may view our square functions S_j, \tilde{S}_j as Hilbert space-valued linear operators

$$\begin{aligned} S_j f(x) &= (T_j^t f(x))_{L^2((0,\infty); dt/t)}, \\ \tilde{S}_j f(x) &= (\tilde{T}_j^t f(x))_{L^2((0,\infty); dt/t)} \end{aligned}$$

so that we may appeal to Lemma 2 for their mapping properties. Take an H^1 -atom a supported in a cube Q .

LEMMA 3. For $j \geq 1$ and an arbitrarily small $\epsilon > 0$,

$$(3-2) \quad \int_{c\bar{Q}} |S_j a(x)| dx \leq C 2^{j(1+\epsilon-\delta)}.$$

Proof. Using the translation invariance of our operator, we may assume the center of Q lies at the origin. By the Cauchy-Schwartz inequality

$$\begin{aligned} \int_{c\bar{Q}} |S_j a(x)| dx &= \int_{c\bar{Q}} |x|^{-l+\epsilon} |x|^{l-\epsilon} |S_j a(x)| dx \\ &\leq \left(\int_{c\bar{Q}} |x|^{-2l+2\epsilon} dx \right)^{1/2} \left(\int_{c\bar{Q}} |x|^{2l-2\epsilon} |S_j a(x)|^2 dx \right)^{1/2} \\ &\leq C |Q|^{\frac{1}{2} + \frac{\epsilon}{n} - \frac{l}{n}} \left(\int_{c\bar{Q}} |x|^{2l-2\epsilon} |S_j a(x)|^2 dx \right)^{1/2}. \end{aligned}$$

Denoting by $l(Q)$ the side-length of Q , we observe that

$$\begin{aligned} \int_{c\bar{Q}} |x|^{2l-2\epsilon} |S_j a(x)|^2 dx &= \int_0^\infty \int_{c\bar{Q}} |x|^{2l-2\epsilon} |T_j^t a(x)|^2 dx \frac{dt}{t} \\ &= \left(\int_0^{l(Q)} + \int_{l(Q)}^\infty \right) \int_{c\bar{Q}} |x|^{2l-2\epsilon} |T_j^t a(x)|^2 dx \frac{dt}{t} \\ &= \text{(I)} + \text{(II)}. \end{aligned}$$

With $\hat{K}_j = m_j$, $K_j^t(x) = t^{-n} K_j(x/t)$, we note that

$$|T_j^t a(x)| = |K_j^t * a(x)| = \|a\|_2 \left(\int_Q |K_j^t(x-y)|^2 dy \right)^{1/2}$$

so

$$\text{(I)} \leq |Q|^{-1} \int_0^{l(Q)} \int_{c\bar{Q}} |x|^{2l-2\epsilon} \int_Q |K_j^t(x-y)|^2 dy dx \frac{dt}{t}.$$

Since $|x - y| \approx |x|$ whenever $x \in {}^c\bar{Q}$, $y \in Q$,

$$\begin{aligned} \text{(I)} &\leq |Q|^{-1} \int_0^{l(Q)} \int_Q \int_{\mathbb{R}^n} |x|^{2l-2\epsilon} |K_j^t(x)|^2 dx dy \frac{dt}{t} \\ &\leq \int_0^{l(Q)} t^{2l-2\epsilon-n} \frac{dt}{t} \int_{\mathbb{R}^n} |x|^{2l-2\epsilon} |K_j(x)|^2 dx \\ &= |Q|^{\frac{2l}{n} - \frac{2\epsilon}{n} - 1} \int_{\mathbb{R}^n} |x|^{2l-2\epsilon} |K_j(x)|^2 dx. \end{aligned}$$

Applying the basic properties of the Fourier transform and the fractional integration theorem of Hardy-Littlewood-Sobolev ([7], pp. 354), we get

$$\begin{aligned} \int_{\mathbb{R}^n} |x|^{2l-2\epsilon} |K_j(x)|^2 dx &= \int_{\mathbb{R}^n} \left| |x|^{-\epsilon} (|x|^l K_j(x)) \right|^2 dx \\ &= C \int_{\mathbb{R}^n} \left| |\xi|^{-n+\epsilon} * \sum_{|\beta|=l} D^\beta m_j(\xi) \right|^2 d\xi \\ &= C \sum_{|\beta|=l} \int_{\mathbb{R}^n} \left| |\xi|^{-n+\epsilon} * D^\beta m_j(\xi) \right|^2 d\xi \\ &\leq C \sum_{|\beta|=l} \left(\int_{\mathbb{R}^n} |D^\beta m_j(\xi)|^p d\xi \right)^{2/p}, \quad 1/p = 1/2 + \epsilon/n. \end{aligned}$$

If we make use of Hölder's inequality, then

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |D^\beta m_j(\xi)|^p d\xi \right)^{2/p} &= \left(\int_{2^{j-1} \leq |\xi| \leq 2^{j+1}} |D^\beta m_j(\xi)|^p d\xi \right)^{2/p} \\ &\leq 2^{2j\epsilon} \int_{2^{j-1} \leq |\xi| \leq 2^{j+1}} |D^\beta m_j(\xi)|^2 d\xi \end{aligned}$$

and therefore by hypothesis (2), we are led to

$$\text{(I)} \leq C |Q|^{\frac{2l}{n} - \frac{2\epsilon}{n} - 1} 2^{2j(\epsilon-\delta)}.$$

For the part (II) we use the vanishing property of atoms to write down

$$\begin{aligned} T_j^t a(x) &= \int_Q K_j^t(x-y) a(y) dy \\ &= \int_Q [K_j^t(x-y) - K_j^t(x)] a(y) dy \\ &= \sum_{|\gamma|=1} \int_Q \int_0^1 (-y)^\gamma D_x^\gamma K_j^t(x-sy) a(y) ds dy, \end{aligned}$$

where the last equality comes from Taylor’s theorem. Hence

$$|T_j^t a(x)| \leq C |Q|^{\frac{1}{n}-\frac{1}{2}} \left(\int_Q \int_0^1 |D_x^\gamma K_j^t(x-sy)| ds dy \right)^{1/2}$$

and it follows that

$$\begin{aligned} \text{(II)} &\leq C |Q|^{\frac{2}{n}-1} \int_{l(Q)}^\infty \int_{c\bar{Q}} |x|^{2l-2\epsilon} \int_Q \int_0^1 |D_x^\gamma K_j^t(x-sy)|^2 ds dy dx \frac{dt}{t} \\ &\leq C |Q|^{\frac{2}{n}} \int_{l(Q)}^\infty t^{2l-2\epsilon-n-2} \int_{\mathbb{R}^n} |x|^{2l-2\epsilon} |D^\gamma K_j(x)|^2 dx \frac{dt}{t} \\ &\leq C |Q|^{\frac{2l}{n}-\frac{2\epsilon}{n}-1} \int_{\mathbb{R}^n} \left| |x|^{-\epsilon} (|x|^l D^\gamma K_j(x)) \right|^2 dx \\ &\leq C |Q|^{\frac{2l}{n}-\frac{2\epsilon}{n}-1} \sum_{|\beta|=l} \left(\int_{\mathbb{R}^n} |D^\beta(\xi^\gamma m_j(\xi))|^p d\xi \right)^{2/p}, \quad 1/p = 1/2 + \epsilon/n \\ &\leq C |Q|^{\frac{2l}{n}-\frac{2\epsilon}{n}-1} 2^{2j\epsilon} \sum_{|\beta|=l} \int_{2^{j-1} \leq |\xi| \leq 2^{j+1}} |D^\beta(\xi^\gamma m_j(\xi))|^2 d\xi \end{aligned}$$

as before. Here recall $|\gamma| \cong 1$ and the hypothesis (2) shows then

$$\text{(II)} \leq C |Q|^{\frac{2l}{n}-\frac{2\epsilon}{n}-1} 2^{2j(1+\epsilon-\delta)}.$$

Combining both estimates, we finish the proof. \square

Now it’s a relatively easy matter to state

LEMMA 4. For $j \geq 1$ and an arbitrarily small $\epsilon > 0$,

$$(3-3) \quad \int_{\epsilon \tilde{Q}} |\tilde{S}_j a(x)| dx \leq C 2^{j(2+\epsilon-\delta)}.$$

Proof of Theorem 1. Since $\|S_j\|_{2,2} \leq 2^{-ja}$, $\|\tilde{S}_j\|_{2,2} \leq 2^{j(1-b)}$, Lemma 2 gives us

$$\begin{aligned} \|S_j f\|_{L^1} &\leq C \left(2^{j(1+\epsilon-\delta)} + 2^{-ja} \right) \|f\|_{H^1}, \\ \|\tilde{S}_j f\|_{L^1} &\leq C \left(2^{j(2+\epsilon-\delta)} + 2^{j(1-b)} \right) \|f\|_{H^1} \end{aligned}$$

and consequently (3-1) shows

$$\|T_j^* f\|_{L^1} \leq C \left(2^{j(\epsilon+\frac{3}{2}-\delta)} + 2^{j(1-a-b)} \right) \|f\|_{H^1}.$$

Upon summing this geometric series, the proof is now completed. \square

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