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CONSTRUCTION OF A COMPLETE NEGATIVELY CURVED SINGULAR RIEMANNIAN FOLIATION

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Let (M, g) be a complete Riemannian manifold and G be a closed (connected) subgroup of the group of isometries of M. Then the union $\overset{\circ}{M}$ of all principal orbits is an open dense subset of M and the quotient map $\overset{\circ}{M} \longrightarrow \overset{\circ}{B} := \overset{\circ}{M}/G$ becomes a Riemannian submersion for the restriction of g to $\overset{\circ}{M}$ which gives the quotient metric on $\overset{\circ}{B}$. Namely, B is a singular (complete) Riemannian space such that ∂B consists of non-principal orbits.

We shall discuss a complete singular Riemannian foliation $\mathcal{F}$ on a complete Riemannian manifold (M, g) of dimension m ([4]) such that $B := M/\mathcal{F}$ is a Riemannian manifold of dimension q with boundary, namely, $\overset{\circ}{B}$ is the regular stratum and ∂B consists of singular leaves.

Terminology "complete" means that M is complete as a metric space. In this situation, we can construct a complete negatively curved singular Riemannian foliation with leaves $S^{m-q}(1)$ from $(M, g, \mathcal{F})$ with the additional conditions :

THEOREM. *Let f be a C^∞-function on B. Suppose that B and f satisfy the following conditions :*

(B.1) *the canonically endowed continuous metric of the Riemannian simple double $2B$ is smooth,*

(B.2) *the sectional curvature K_B of B is negative, namely, the transversal sectional curvature of $\mathcal{F}$ is negative, or $B := [0, \infty)$,*

(F.1) *f is a function of the geodesic distance r from ∂B, namely, f is a basic function of the transversal distance r,*

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(F.2) *f is an odd function of r on a neighborhood of $r = 0$ and satisfies that $f'(0) = 1$, $f''(r) > 0$ for $r > 0$, and $f'''(r) > 0$.*

Then there exists a complete strictly negatively curved Riemannian manifold with $S^{m-q}(1)$ as generic leaves.

We shall be in C^∞ -category and manifolds are assumed to be connected, paracompact, Hausdorff spaces.

1. Preliminaries

LEMMA 1.1. (cf. [3], [5], p31) *If $f(t)$ is a real-valued C^∞-even function on $\mathbf{R}$, then $f(r)$ is a C^∞-function on $\mathbf{R}^{m-q}$, where $r := ((x^1)^2 + \cdots + (x^{m-q})^2)^{1/2}$.*

LEMMA 1.2. ([2]) *Let $f : \mathbf{R}^q \times \mathbf{R}^{m-q+1} \longrightarrow \mathbf{R}$ be a continuous function. If f satisfies the following conditions :*

(1.2.1) f is of class C^∞ on $(\mathbf{R}^q \times \mathbf{R}^{m-q+1})\backslash(\mathbf{R}^q \times \{0\})$,

(1.2.2) f is invariant under $\{I_q\} \times O(m-q+1)$, where I_q is the unit group on $\mathbf{R}^q$ and $O(m-q+1)$ is the rotation group on $\mathbf{R}^{m-q+1}$,

(1.2.3) f is of class C^∞ on $\mathbf{R}^q \times l$ for any straight line $l \subset \mathbf{R}^{m-q+1}$ through the origin, then f is of class C^∞ on $\mathbf{R}^q \times \mathbf{R}^{m-q+1}$.

We suppose that B and f satisfy the following conditions :

(1) B has the Riemannian simple double $2B$,

(2) $f(x) > 0$ if $x \in B\backslash\partial B$, and f is an odd function on a neighborhood of ∂B of the arc-length r in the inner normal direction to ∂B,

(3) $\|grad\ f\|(x) = 1$ if $x \in \partial B$.

Let (U, ϕ) be a local patch of ∂B around a singular leaf whose dimension is less than that of the generic leaf. Let N be the ϵ-collar neighborhood of U in B. We define a manifold $\mathcal{N}$ by

$$\mathcal{N} := (N\backslash U) \times_{f|_{N\backslash U}} S^{m-q}(1).$$

Imbedding of $S^{m-q}(1)$ into $\mathbf{R}^{m-q+1}$, we define a diffeomorphism Ψ of $\mathcal{N}$ into $\mathbf{R}^q \times \mathbf{R}^{m-q+1}$ by

$$\Psi : ((x, \exp rX), y) \to (\phi(x), ry),$$

where $X \in T_xB$ is the unit inner normal vector to ∂B and $0 < r < \epsilon$.

We take the Riemannian metric g' on $\Psi(\mathcal{N})$ so that Ψ may become an isometry. Note that g' can be extended to the continuous metric $\overline{g'}$ on $\overline{\Psi(\mathcal{N})}$ which is the closure of $\Psi(\mathcal{N})$ by the natural way. We have only to show that $\overline{g'}$ is of class C^∞ at the origin. Let $(x^1, \cdots, x^q, x^{q+1}, \cdots, x^{m+1})$ be the Cartesian coordinates of $\mathbf{R}^q \times \mathbf{R}^{m-q+1}$. And we adopt the ranges of indices :

$$1 \leq i, j \leq q \quad \text{and} \quad q+1 \leq \alpha, \beta \leq m+1.$$

It is clear from Lemma 1.2 that $\overline{g'}_{ij} := \overline{g'}(\partial/\partial x^i, \partial/\partial x^j)$ is of class C^∞. Moreover, we have $\overline{g'}_{i\alpha} := \overline{g'}(\partial/\partial x^i, \partial/\partial x^\alpha) = x^\alpha(1/r)\overline{g'}(\partial/\partial x^i, \partial/\partial r)$ is of class C^∞. Finally we see that using polar coordinates

$$\begin{aligned}
\overline{g'}_{\alpha\beta} &:= \overline{g'}(\partial/\partial x^\alpha, \partial/\partial x^\beta) \\
&= \tilde{g}_{\alpha\beta} + \frac{f^2(x,r) - r^2}{r^4} r^4 g_{S^{m-q}}(\partial/\partial x^\alpha, \partial/\partial x^\beta) \\
&= \tilde{g}_{\alpha\beta} + \frac{f^2(x,r) - r^2}{r^4} (r^2 \tilde{g}_{\alpha\beta} - x^\alpha x^\beta)
\end{aligned}$$

where $\tilde{g}$ is the standard metric on $\mathbf{R}^q \times \mathbf{R}^{m-q+1}$. It follows from Lemma 1.2 that $\frac{f^2(x,r)-r^2}{r^4}$ is of class C^∞. Therefore, $\overline{g'}_{\alpha\beta}$ is of class C^∞.

Taking $S^{m-q}(1)$ in the tangent space to the generic leaf, we set $\tilde{M} := (B \backslash \partial B) \times_{f|_{B\backslash\partial B}} S^{m-q}(1)$. Then there exists the unique complete singular Riemannian foliation on $\mathcal{M}$ with $S^{m-q}(1)$ as leaves which is the completion of $\tilde{M}$.

Summing up, we have

PROPOSITION 1.3. *Let $\mathring{M}$ be the regular stratum and $\mathring{B} := \mathring{M}/\mathcal{F}$. Suppose that B and f satisfy the following conditions :*

(1) *B has the Riemannian simple double $2B$,*
(2) *$f(x) > 0$ if $x \in B \backslash \partial B$, and f is an odd function on a neighborhood of ∂B of the arc-length r in the inner normal direction to ∂B,*
(3) *$||grad\ f||(x) = 1$ if $x \in \partial B$.*

Then there exists the unique complete singular Riemannian foliation on $\mathcal{M}$ with $S^{m-q}(1)$ as generic leaves.

LEMMA 1.4. ([1]) *Let $M := B \times_f F$ be a warped product with a warping function f where B and F are any Riemannian manifolds. Let π_1 and π_2 be the natural projections respectively. Let Π be a 2-plane tangent to M at x and $\{X+V, Y+W\}$ an orthonormal basis for Π, where $X, Y \in T_{\pi_1(x)}B$ and $V, W \in T_{\pi_2(x)}F$. The sectional curvature $K(\Pi)$ of Π in M is given by*

$$K(\Pi) = K^1_{X,Y} + K^2_{X,Y,V,W} + K^3_{V,W},$$

where

$$\begin{aligned} K^1_{X,Y} &:= K_B(X,Y)\|X \wedge Y\|^2_B \\ K^2_{X,Y,V,W} &:= -f(\pi_1(x))\{\|W\|^2_F((\nabla_B)^2 f)(X,X) \\ &\quad - 2 < V, W >_F ((\nabla_B)^2 f)(X,Y) + \|V\|^2_F((\nabla_B)^2 f)(Y,Y)\}, \\ K^3_{V,W} &:= f^2(\pi(x))\{K_F(V,W) - \|grad\ f\|^2_B\}\|V \wedge W\|^2_F, \end{aligned}$$

and $\nabla_{(\cdot)}$ and $K_{(\cdot)}$ denote the covariant derivative and the sectional curvature of $(\cdot)$ respectively and $(\nabla_B)^2 f$ denotes the Hessian of f.

2. Proof of Theorem

By the conditions imposed on B, there is a diffeomorphism $\Psi : \partial B \times [0,\infty) \longrightarrow B$ such that, for any $x \in \partial B, \tau_x(r) := \Psi(x,r)$ is the geodesic parametrized by the arc-length r, starting at x and normal to ∂B. Since Lemma 2.1, $(B.2)$ and $(F.2)$ imply that K^1, K^2 and K^3 are non-positive on M and at least one of K^1, K^2 and K^3 is strictly negative, it is enough to show that at least one of K^1, K^2 and K^3 is strictly negative if $r \to 0$.

Let x_0 be a boundary point of B and X_r, Y_r, V_r, W_r be any vector fields along $\tau_{x_0}(r)$, where X_r, Y_r are transversal and V_r, W_r are tangent to leaves if $r \neq 0$.

Case 1. The case that X_0 and Y_0 are linearly independent. We have

$$K^1_{X_0,Y_0} < 0.$$

Case 2. The case that V_0 and W_0 are linearly independent. $(F.1)$ and $(F.2)$ imply that

$$f^2(r) = r^2 + 2ar^4 + \cdots, \quad a > 0$$

and

$$
\begin{aligned}
||grad\ f(r)||_B^2 &\geq < grad\ f(r), \partial/\partial r >_B^2 \\
&= \left(\frac{\partial f(r)}{\partial r}\right)^2 \\
&= 1 + 6ar^2 + \cdots .
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\frac{1 - ||grad\ f(r)||_B^2}{f^2(r)} &\leq \frac{1 - (1 + 6ar^2 + \cdots)}{r^2 + 2ar^4 + \cdots} \\
&= \frac{-6a + o(r)}{1 + o(r)},
\end{aligned}
$$

so that

$$
\lim_{r \to 0} K^3_{V_r, W_r} \leq -6a < 0.
$$

Case 3. The case except Case 1 and Case 2. We can choose X_r, Y_r, V_r, W_r such that $Y_r = c_1 X_r$ and $W_r = c_2 V_r$, where c_1 and c_2 are constants with $c_1 \neq c_2$. Let Π_r be the 2-plane spanned by the orthonormal basis $\{X_r + V_r, Y_r + W_r\}$. Then we have

$$
K(\Pi_r) = -\frac{((\nabla_B)^2 f)(X_r, X_r)}{f(r) < X_r, X_r >_B}.
$$

To get $\lim_{r \to 0} K(\Pi_r) < 0$, it is enough to show that

$$
\lim_{r \to 0} \frac{((\nabla_B)^2 f)(X_r, X_r)}{f(r)} > 0
$$

under the assumption $||X_r||B = 1$.

$$
\frac{((\nabla_B)^2 f)(X_r, X_r)}{f(r)} = \frac{f''(r)(\nabla_{B X_r} r)^2 + f'(r)((\nabla_B)^2 r)(X_r, X_r)}{f(r)},
$$

and $(F.2)$ imply the claim. Therefore we have the Theorem.

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