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ON THE ASYMPTOTIC–NORMING PROPERTY AND THE MAZUR INTERSECTION PROPERTY

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1. Introduction

Unless otherwise stated, we always assume that X is a Banach space, and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. We use $S(X)$ and $B(X)$ to denote the unit sphere and the unit ball in X respectively.

The asymptotic-norming property(ANP) was introduced by James and Ho [12] to demonstrate that there is a larger class of Banach spaces that satisfy the Radon-Nikodym property(RNP) than the class of Banach spaces that are isomorphic to subspaces of separable dual. Ghoussoub and Maurey [8] proved that for separable Banach spaces the ANP is equivalent to the RNP. Hu and Lin [10] studied the three asymptotic-norming properties are equivalent in a larger class of Banach space than separable Banach spaces. However, in general, it is an open question whether the two properties are equivalent. In this paper, we prove that the three asymptotic-norming properties are equivalent in Banach lattices. Also we prove that if X is an order continuous Banach lattice, then for dual X^* the w^*-ANP is equivalent to the RNP.

Giles, Gregory and Sims [9] proved that X has the Mazur intersection property(MIP) if and only if the weak* denting points of $B(X^*)$ are norm dense in $S(X^*)$. And they proved that X^* has the w^*-MIP if and only if the denting points of $B(X)$ are norm dense in $S(X)$. Bandyopadyaya and Roy [1] proved that if μ denotes the Lebesgue measure on $[0, 1]$ and $1 < p < \infty$, then $L_p(\mu, X)$ has the MIP if and only if X has the MIP and is an Asplund space. In this paper, we

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prove that X^* has the w^*-MIP if and only if $E(X)^*$ has the w^*-MIP. This is an extension of the result in [3].

A set Φ in X^* is a norming set for X if $\Phi \subset B(X^*)$ and $\| x \| = \sup_{x^* \in \Phi} x^*(x)$ for all x in X. A sequence $\{x_n\}$ in $S(X)$ is said to be asymptotically normed by for any $\varepsilon > 0$, there is $x^* \in \Phi$ and $m \in \mathbb{N}$ such that $x^*(x_n) > 1 - \varepsilon$ for all $n \geq m$. We say that X has the Φ-ANP-I(resp. Φ-ANP-II; or Φ-ANP-III) if every sequence $\{x_n\}$ in $S(X)$ that is asymptotically normed by Φ is convergent (resp. has a convergent subsequence; or $\bigcap_{n=1}^{\infty} \overline{co}\{x_k \mid k \geq n\} \neq \emptyset$). And we say that $(X, \| \cdot \|)$ has the ANP-k for k=I, II or III if there is a norming set Φ for $(X, \| \cdot \|)$ such that X has the Φ-ANP-k. The space X has the ANP-k for k=I, II or III if there is an equivalent norm $\| \cdot \|$ on X such that $(X, \| \cdot \|)$ has the ANP-k. X^* is said to have the *weak** asymptotic-norming property k(w^* -ANP-k) for k=I, II or III, if there is an equivalent norm $\| \cdot \|$ on X and a norming set Φ in $B((X, \| \cdot \|))$ such that $(X^*, \| \cdot \|)$ has the Φ-ANP-k for k=I, II or III.

As was shown in [10], the ANP is related to property (G) which can be described in terms of denting points. An element $x \in S(X)$ is a denting point of $B(X)$ if $x \notin \overline{co}(M(x, \varepsilon))$ for all $\varepsilon > 0$ where $M(x, \varepsilon) = \{y \in X : \| y \| \leq 1 \textit{ and } \| x - y \| > \varepsilon\}$. A Banach space X is said to have property (G) if every point of $S(X)$ is a denting point of $B(X)$. We use $DentK$ to denote the set of denting points of K. An element $x^* \in S(X^*)$ is a *weak** denting point of $B(X^*)$ if for every $\varepsilon > 0$ there are an element x in X and $\delta > 0$ such that diameter of $S(x^*, x, \delta) = \{y^* : \ y^* \in B(X^*), y^*(x) > x^*(x) - \delta\}$ is less than ε. X^* is said to have property (G^*) if every point of $S(X^*)$ is a weak* denting point of $B(X^*)$.

A Banach space X has the Mazur intersection property(MIP) if every bounded closed convex set in X can be represented as an intersection of balls in X. A dual Banach space X^* has the weak*-Mazur intersection property(w*-MIP) if every bounded weak*-closed convex set in X^* can be represented as an intersection of balls in X^*.

A norm $\| \cdot \|$ on a Banach space X is locally uniformly convex (LUR) if for each $\epsilon > 0$ and each $x \in S(X)$, there exists $\delta(x, \epsilon) > 0$ such that if $y \in S(X)$ and $\| x - y \| \geq \epsilon$, then $\| \frac{x+y}{2} \| < 1 - \delta(x, \epsilon)$ [5].

A Banach lattice X is order continuous if, for every downward directed set $\{x_\alpha\}$ in X with $\inf\{x_\alpha\} = 0$, $\lim \| x_\alpha \| = 0$. David, Ghoussoub and Lindenstrauss [4] proved that a Banach lattice X is order continuous if and only if it has an equivalent locally uniformly convex lattice norm. And Ghoussoub [7] proved that the following: If X is an order continuous Banach lattice and if X^* is order continuous, then there exists an equivalent lattice norm on X which is LUR such that its dual norm is also LUR on X^*.

2. Asymptotic-norming and Radon-Nikodym property

In this section, we prove that the three asymptotic-norming properties are equivalent in Banach lattices and that if X is an order continuous Banach lattice, then X^* has the RNP if and only if X^* has the w^*-ANP-I.

LEMMA 2.1. *Let X be an order continuous Banach lattice. Then there is an equivalent lattice norm $\|\cdot\|$ such that $(X, \|\cdot\|)$ has the property (G).*

Proof. Let X be an order continuous Banach lattice. Then it has an equivalent LUR lattice norm $\|\cdot\|$ [4]. Since $(X, \|\cdot\|)$ is LUR, $(X, \|\cdot\|)$ has the property (G).

THEOREM 2.2. *Let X be a Banach lattice. Then the following are equivalent:*
(1) *X has the ANP-I,*
(2) *X has the ANP-II,*
(3) *X has the ANP-III.*

Proof. Suppose X has the ANP-III. Then X has the RNP [12], and then X has the the Krein Miliman Property. Thus c_o is not lattice isomorphic to a sublattice of X and without loss of generality we may assume that X is order continuous. Then by Lemma 2.1 there exists an equivalent lattice norm $\|\cdot\|$ on X such that $(X, \|\cdot\|)$ has the property (G). Thus by Theorem 2.7 in [10] X has the ANP-I.

COROLLARY 2.3. *Let X be a Banach lattice. Then the following are equivalent:*
(1) *$L_p(\mu, X)$ has the ANP-I,*
(2) *$L_p(\mu, X)$ has the ANP-II,*

(3) *$L_p(\mu, X)$ has the ANP-III.*

Proof. The proof follows from Theorem 2.2 and Theorem 6 in [11].

THEOREM 2.4. *Let X be an order continuous Banach lattice. If X^* is order continuous, then there exists an equivalent lattice norm $\|\cdot\|$ on X such that $(X^*, \|\cdot\|)$ has the property (G^*).*

Proof. Since X and X^* is order continuous, there exists an equivalent lattice norm $\|\cdot\|$ on X such that its dual norm is LUR on X^* [7]. Let $h \in S(X^*)$. Since $\|\cdot\|$ is LUR, for arbitrary $\epsilon > 0$, there exists an $\delta > 0$ such that $f \in B(X^*)$ and $\| \frac{f+g}{2} \| > 1 - \delta$, then $\| f - h \| < \epsilon$. Choose an $x \in S(X)$ so that $h(x) > 1 - \frac{\delta}{2}$. Then if $f \in B(X^*)$ and $f(x) > 1 - \frac{\delta}{2}$, then $\| \frac{f+h}{2} \| \geq \frac{f+h}{2}(x) > 1 - \delta$ and thus by local uniform convexity of $\|\cdot\|$ at h, $\| f - h \| < \epsilon$. Therefore h is a weak*-denting point of $B(X^*)$. Hence $(X^*, \|\cdot\|)$ has the property (G^*).

COROLLARY 2.5. *Let X be an order continuous Banach lattice. Then the following are equivalent:*

(1) *X^* has the RNP,*
(2) *X^* has the w^*-ANP-I,*
(3) *X^* has the w^*-ANP-II,*
(4) *X^* has the w^*-ANP-III.*

Proof. If X^* has the w^*-ANP-III, then X^* has the RNP[10]. Conversely if X^* has the RNP, then no closed sublattice of X^* is lattice isomorphic to either c_0 or $L_1[0,1]$ [14] and thus X^* is order continuous. By Theorem 2.4 there exists an equivalent lattice norm $\|\cdot\|$ on X such that $(X^*, \|\cdot\|)$ has the property (G^*). And thus $(X^*, \|\cdot\|)$ has the Φ-ANP-I for some norming set Φ of $(X^*, \|\cdot\|)$ in $B(X, \|\cdot\|)$. Hence X^* has the w^*-ANP-I.

REMARK. On the above Corollary the assumption that X be order continuous is necessary.

EXAMPLE 2.6. Let $X = C([0, \Omega])$, where Ω is the first uncountable ordinal. Then X is not order continuous. Since X is an Asplund space, X^* has the RNP but X^* fails to have the w^*-ANP-III [10].

COROLLARY 2.7. *Let X be a reflexive Banach lattice. Then the following are equivalent:*

(1) *$L_p(\mu, X)^*$ has the w^*-ANP-I,*

(2) *$L_p(\mu, X)^*$ has the w^*-ANP-II,*

(3) *$L_p(\mu, X)^*$ has the w^*-ANP-III.*

Proof. Since X is reflexive, $L_p(\mu, X)$ is reflexive [6] and thus $L_p(\mu, X)^*$ has the RNP. So $L_p(\mu, X)$ is order continuous [15]. Hence the proof follows by Corollary 2.5.

3. Mazur intersection properties in Köthe-Bochner function spaces

In this section we consider the stability of the weak*-Mazur intersection property. Let (Ω, Σ, μ) denote a measure space with a σ-finite and complete measure μ and $L^o = L^o(\Omega)$ the space of μ-equivalence classes of Σ-measurable real-valued functions. The notation $f \leq g$ for $f, g \in L^o$ will means that $f(t) \leq g(t)$ μ-a.e. in Ω.

DEFINITION 3.1 15]. A Banach space $(E, \|\cdot\|_E) \subset L^o$ is said to be a Köthe function space if

(i) $|f| \leq |g|$, $f \in L^o$, $g \in E$ imply $f \in E$ and $\|f\|_E \leq \|g\|_E$,

(ii) $supp\ E = \bigcup\{supp\ f : f \in E\} = \Omega$.

Every Köthe function space is a Banach lattice in the obvious order ($f \geq 0$ if $f(\omega) \geq 0$ a.e.) [13].

Now let us define the type of spaces to be considered in this section. For a real Banach space $(X, \|\cdot\|_X)$, denote $\mu(X)$, the family of all strongly measurable functions $f : \Omega \longrightarrow X$ identifying functions which are μ-a.e. equal. Let

$$E(X) = \{f \in \mu(X) : \|f(\cdot)\|_X \in E\}.$$

Then $E(X)$ becomes a Banach space with the norm $\|f\| = \|\|f(\cdot)\|_X\|_E$ and it is called a Köthe-Bochner space.

THEOREM 3.2 [9]. *A dual Banach space X^* has the w^*-MIP if and only if $DentB(X)$ is dense in $S(X)$.*

THEOREM 3.3. *Let E be a Köthe function sapce over a measure space (Ω, Σ, μ) and let X be a Banach space. If X^* has the w^*-MIP ,then $E(X)^*$ has the w^*-MIP.*

Proof. Suppose $E(X)^*$ does not have the w^*-MIP. Then for each $g \in DentB(E(X))$, there exist $f \in S(E(X))$ and $\epsilon > 0$ such that $\| g - f \| \geq \epsilon$. Since $g \in DentB(E(X))$, $\frac{g(t)}{\|g(t)\|} \in DentB(X)$ for μ-a.e. $t \in supp\ g$ [2]. Let

$$B = \{t \in supp\ g : \| f(t) - g(t) \| \geq \frac{\epsilon}{4} max\{\| f(t) \|, \| g(t) \|\}\}.$$

Then

$$\begin{aligned}
\| (f-g)\chi_{\Omega \setminus B} \| &= \| \| (f(\cdot) - g(\cdot))\chi_{\Omega \setminus B} \|_X \|_E \\
&\leq \| \frac{\epsilon}{4} max(\| f(t) \|_X, \| g(t) \|_X) \|_E \\
&\leq \frac{\epsilon}{4} \| \| f(t) \|_X + \| g(t) \|_X \|_E \\
&\leq \frac{\epsilon}{4} (\| f \| + \| g \|) \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

Since $\| (f-g)\chi_{\Omega \setminus B} \| \leq \frac{\epsilon}{2}$ and $\| f - g \| \geq \epsilon$, $\| (f-g)\chi_B \| \geq \frac{\epsilon}{2}$ and therefore $\mu(B) > 0$. Partition the set B into four sets:

$$B_1 = \{t \in B \mid \| g(t) \| < (1 - \frac{\epsilon}{8}) \| f(t) \|\}$$

$$B_2 = \{t \in B \mid (1 - \frac{\epsilon}{8}) \| f(t) \| \leq \| g(t) \| \leq \| f(t) \|\}$$

$$B_3 = \{t \in B \mid (1 - \frac{\epsilon}{8}) \| g(t) \| \leq \| f(t) \| \leq \| g(t) \|\}$$

$$B_4 = \{t \in B \mid \| f(t) \| < (1 - \frac{\epsilon}{8}) \| g(t) \|\}.$$

Then $B_2 \neq \emptyset$ or $B_3 \neq \emptyset$ and $f(t) \neq 0$ for $t \in B_2 \bigcup B_3$. Let $A = \| \frac{f(t)}{\|f(t)\|} - \frac{g(t)}{\|g(t)\|} \|$. If $t \in B_2$, then

$$\begin{aligned}
\alpha &\geq \| \frac{g(t)}{\| f(t) \|} - \frac{f(t)}{\| f(t) \|} \| - \| \frac{g(t)}{\| f(t) \|} - \frac{g(t)}{\| g(t) \|} \| \\
&\geq \frac{\epsilon}{4} - \frac{\| f(t) \| - \| g(t) \|}{\| f(t) \|} \\
&\geq \frac{\epsilon}{8}.
\end{aligned}$$

If $t \in B_3$, then

$$\begin{aligned}\alpha &\geq \left\| \frac{f(t)}{\| g(t) \|} - \frac{g(t)}{\| g(t) \|} \right\| - \left\| \frac{f(t)}{\| g(t) \|} - \frac{f(t)}{\| f(t) \|} \right\| \\ &\geq \frac{\epsilon}{4} - \frac{\| g(t) \| - \| f(t) \|}{\| g(t) \|} \\ &\geq \frac{\epsilon}{8}.\end{aligned}$$

Thus $\alpha \geq \frac{\epsilon}{8}$ for some $t \in B$. Therefore $\frac{f(t)}{\|f(t)\|} \notin \overline{DentB(X)}$. But $\frac{f(t)}{\|f(t)\|} \in S(X)$. Hence X^* does not have the w^*-MIP by Theorem 3.2.

THEOREM 3.4. *Let E be an order continuous Köthe function space over a measure space (Ω, Σ, μ) and let X be a Banach space. If $E(X)^*$ has the w^*-MIP, then X^* has the w^*-MIP.*

Proof. Suppose $E(X)^*$ has the w^*-MIP. Then $\overline{DentB(E(X))} = S(E(X))$. First assume that $0 < \mu(\Omega) < \infty$. Let $x \in S(X)$ and choose $B \in \Sigma$ so that $\mu(B) > 0$ and $\| x\chi_B \| = 1$. Hence for each $\epsilon (0 < \epsilon < 1)$, there exists $f \in DentB(E(X))$ such that $\| x\chi_B - f \|_{E(X)} < \epsilon$. Let $A \subset B$ be given by $\{t \mid t \in B \text{ and } \| x - f(t) \|_X < \epsilon\}$. Then $A \subset supp\, f$. Since

$$\begin{aligned}\| f\chi_A \| &= \| \| f(\cdot)\chi_A \|_X \|_E \\ &\geq \| \| x \|_X - \| x - f(\cdot)\chi_A \|_X \|_E \\ &\geq \| x\chi_\Omega \|_{E(X)} - \| x\chi_\Omega - f\chi_A \|_{E(X)} \\ &\geq 1 - \epsilon > 0,\end{aligned}$$

$\mu(A) > 0$. Since $f \in DentB(E(X))$, $\frac{f(t)}{\|f(t)\|} \in DentB(X)$ for a.e. $t \in supp\, f$ [2] and hence there exists $t \in A$ such that $\frac{f(t)}{\|f(t)\|} \in DentB(X)$. Since $\| x \| = 1$,

$$\begin{aligned}\left\| \frac{f(t)}{\| f(t) \|} - x \right\| &\leq \left\| \frac{f(t)}{\| f(t) \|} - f(t) \right\| + \| f(t) - x \| \\ &< | 1 - \| f(t) \| | + \epsilon \\ &\leq \| x - f(t) \| + \epsilon \\ &\leq 2\epsilon.\end{aligned}$$

Thus $DentB(X)$ is dense in $S(X)$. Secondly if Ω is not of finite measure, let $\Omega = \bigcup_n \Omega_n$ with $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \cdots$ and $\mu(\Omega_n) < \infty$. Let $x \in S(X)$. Then $\| x\chi_{\Omega_n}(\cdot) \|_X \nearrow \| x \|_X = 1$. Since E is order continuous [4], for arbitrary $\epsilon > 0$ there exists $g \in S(E(X))$ such that $\| x\chi_{\Omega_n} - g \|_{E(X)} < \frac{\epsilon}{2}$ for sufficiently large n [13]. Since $\overline{DentB(E(X))} = S(E(X))$, for each $\epsilon(0 < \epsilon < 1)$, there exists $f \in DentB(E(X))$ such that $\| g - f \|_{E(X)} < \frac{\epsilon}{2}$. Thus $\| x\chi_{\Omega_n} - f \|_{E(X)} < \epsilon$ for sufficiently large n. Since $\mu(\Omega_n) < \infty$ for all n, $DentB(X)$ is dense in $S(X)$. Hence X^* has the w^*-MIP by Theorem 3.2.

COROLLARY 3.5 [3]. *Let X be any Banach space. X^* has the w^*-MIP if and only if $L_p(\mu, X)^*$ has the w^*-MIP.*

Proof. Since $L_p(\mu, X)$ is uniformly convex, the result is implied by Theorems 3.3 and 3.4.

References

1. P. Bandyopadhyaya and A. K. Roy, *Some satbility results for Banach spaces with the Mazur intersection property*, Indag. Math., N.S. **1(2)** (1990), 137-154.
2. C. Castaing and R. Pluciennik, *Denting points in Köthe-Bochner sapces*, Set-Valued Analysis **2** (1994), 439-458.
3. S. J. Cho, *Mazur intersection properties in Lebesgue-Bochner function spaces*, submitted.
4. W. J. Davis, N. Ghoussoub and J. Lindenstrauss, *A lattice renorming theorem and application to vector-valued processess*, Trans. Amer. Math. Soc. **263** (1981), 531-540.
5. J. Diestel, *Geometry of Banach spaces selected topics*, Lect. Notes in Math., Springer-Verlag **485** (1975).
6. J. Diestel and J. J. Uhl, Jr., *Vector measures*, Math. Survey, Amer. Math. Soc. **15** (1977).
7. N. Ghoussoub, *Renorming dual Banach lattices*, Proc. Amer. Math. Soc. **84** (1982), 521-524.
8. N. Ghoussoub and B. Maurey, *The asymptotic-norming and the Radon-Nikodym properties are equivalent in separable Banach spaces*, Proc. Amer. Math. Soc. **94** (1985), 665-671.
9. J. R. Giles, D. A. Gregory and B. Sims, *Characterization of normed linear spaces with Mazur's intersection property*, Bull. Austral. Math. Soc. **18** (1978), 105-123.
10. Z. Hu and B. L. Lin, *On the asymptotic-norming property of Banach spaces, Function spaces*, Lecture Notes in Pure and Applied Math. **136** (1992); Marcel Dekker, 195-210.

11. Z. Hu and B. L. Lin, *Asymptotic-norming and Mazur intersection properties in Lebesgue-Bochner function spaces*, Bull. Austral. Math. Soc. **48** (1993), 177-186.
12. R. C. James and A. Ho, *The asymptotic-norming and Radon-Nikodym properties for Banach spaces*, Ark. Mat. **19** (1981), 53-70.
13. L. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II, Function spaces*, Springer-Verlag, New York, 1979.
14. H. P. Lotz, *The Radon-Nikodym property in Banach-lattices (unpublished)*.
15. H. H. Schaefer, *Banach lattices and positive operators*, Berin, Heidelberg, New York: Springer-Verlag, 1978.

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