J. Korean Math. Soc. **32** (1995), No. 3, pp. 541–551

NUMERICAL SOLUTION FOR NONLINEAR KLEIN–GORDON EQUATION BY COLLOCATION METHOD WITH RESPECT TO SPECTRAL METHOD

IN JUNG LEE

1. Introduction

The nonlinear Klein Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + V_u(u) = f \tag{1}$$

where Δ is the Laplacian operator in R^d $(d = 1,2,3)$, $V_u(u)$ is the derivative of the "potential function" V, and f is a source term independent of the solution u, in various areas of mathematical physics. Among the particular cases which are the practical relevance, we take $V_u(u) = |u|^\alpha u$ with $\alpha > 0$ (quantum mechanics), refer to [5].

The convergence of the Galerkin finite element method for second order hyperbolic equations has been studied by many authors: cf. among others Dupont[3], who obtained error estimates for time-discrete and time continuous approximations of linear problems, and Dendy [2], who examined nonlinear problems as well as various modified Galerkin methods. To compute the nonlinear term $V_u(u)$, the product approximation is used by Yves Tourigny [6]. This approximation is a technique which consists of replacing the nonlinear term by its interpolant in the finite-dimensional subspaces. This provides an interesting alternative to numerical quadrature and greatly eases the implementation of the Galerkin method.

In this paper, by the collocation method, when $n = 2$, we get the numerical solutions of (1). We shall show the stability and convergence for using spectral method technic.

Received June 21, 1994. Revised November 18, 1994.
1991 AMS Subject Classification: 65M70.
Key words: collocation method, spectral method.

To define a collocation method with spectral method techics, we give as many distinct points

$$x_k \quad k \in J \quad \text{(a set of indices)}$$

in the domain Ω or in its boundary $\partial\Omega$ as the dimension of the space $Pol_N(\Omega)$ in which the spectral solution is sought. At the number of these points, located on $\partial\Omega$ the boundary conditions are imposed. The remaining points are used to enforce the differential equation.

We assume that for any $k \in J$, there exists a polynomial $\phi_k \in Pol_N(\Omega)$, nessarily unique, such that

$$\phi_k(x_m) = \begin{cases} 1 & \text{if } k = m, \\ 0 & \text{if } k \neq m. \end{cases}$$

The ϕ_k's form a basis for the polynomials of degree N, since $v(x) = \sum_{k \in J} v(x_k)\phi_k(x)$ for all $v \in Pol_N(\Omega)$. Let J be divided into two disjoint subsets J_e and J_b, such that if $k \in J_b$, the x_k's are on the part $\partial\Omega$ of the boundary. Moreover, let L_N be an approximation to the operator L in which derivatives are taken via collocation at the points x_k's. The collocation soluton is a polynomial $u^N \in Pol_N(\Omega)$ which satisfies the equations

$$\begin{cases} L_N u^N(x_k) = f(x_k) & \text{for all } k \in J_e, \\ Bu^N(x_k) = 0 & \text{for all } k \in J_b. \end{cases}$$

The unknowns in a collocation method are the values of u^N at the points x_k, i.e., the coefficients of u^N with respect to the Lagrange basis. We introduce a bilinear form $(u, v)_N$ on the space $Z = C^0(\Omega)$ of the functions continuous up to the boundary of Ω by fixing a family of weights $w_k > 0$ and setting

$$(u, v) = \sum_{k \in J} u(x_k)\overline{v(x_k)}w_k$$

The existence of the Lagrange basis ensures that $(u, v)_N$ is an inner product on $Pol_N(\Omega)$. Consequently, we define a **discrete norm** on $Pol_N(\Omega)$ as

$$\|u\|_N = \{(u, u)_N\}^{1/2} \qquad \text{for } \ u \in Pol_N(\Omega)$$

The basis of the ϕ_k's is orthogonal under the discrete inner product. We make the assumption that the nodes $\{x_k\}$ and the weights $\{w_k\}$ are such that

$$(u,v)_N = (u,v) \text{ for all } u,v \text{ such that } uv \in Pol_{2N-1}(\Omega).$$

In all the applicatios, this assumption is fulfilled since the x_k's are the knots of quadrature formulas of Gaussian type.

Let X_N be the space of the polynomials of degree less than or equal to N which satisfy the boundry conditions, i.e.,

$$X_N = \{v \in Pol_N(\Omega) | Bv(x_k) = 0 \text{ for all } k \in J_b\}$$

Then the collocation method is equivalently written as

$$\begin{cases} u^N \in X_N \\ & \text{for all } k \in J_e. \\ (L_N u^N, \phi_k) = (f, \phi_k)_N \end{cases}$$

If Y_N is the space spanned by the ϕ_k's with $k \in J_e$, i.e.,

$$Y_N = \{v \in Pol_N(\Omega) | v(x_k) = 0 \text{ for all } k \in J_b\}$$

then can be written as

$$\begin{cases} u^N \in X_N \\ (L_N u^N, v)_N = (f, v)_N \text{ for all } v \in Y_N. \end{cases}$$

2. Stability

Let Ω be an interval $[-1,1]$. We would like to approximate the solution of the following problem

$$\tag{2} \begin{gathered} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + |u|^\alpha u = f \text{ in } Q = \Omega \times [0,T] \\ u(.,0) = u_0 \text{ and } \left(\frac{\partial u}{\partial t}\right)(.,0) = u_1 \text{ in } \Omega \\ u(-1,t) = u(1,t) = 0 \qquad \text{for} \qquad t \in [0,T] \end{gathered}$$

where $u_0 \in H_0(\Omega)$, $u_1 \in L(Q)$ and $f \in L(Q)$ are given functions, $T > 0$.

The solution $u^N(x,t)$ of the Legendre Tau approximation of this problem is for all $t > 0$ a polynomial of degree N in x, which is zero at $x = \pm 1$ and satisfies the equations,

$$\begin{aligned}
&\int_{-1}^{1} \left[u_{tt}^N(x,t) - u_{xx}^N(x,t) + |u^N(x,t)|^\alpha u^N(x,t)\right] v(x)dx \\
&= \int_{-1}^{1} f(x,t)v(x)dx \qquad t > 0, \quad \text{for all} \quad t \in P_{N-2} \\
(3) \qquad &\int_{-1}^{1} \left[u^N(x,0) - u_0(x)\right] v(x)dx = 0 \\
&\int_{-1}^{1} \left[u_t^N(x,0) - u_1(x)\right] v(x)dx = 0
\end{aligned}$$

Let we set $X_N = \{u \in P_N | u(-1) = u(1) = 0,\ Y_N = P_{N-2}$ and $(u,v) = \int_{-1}^{1} u(x)v(x)dx$. For all $u \in X_N$ we have

$$-\int_{-1}^{1} u_{xx} P_{N-2} u dx = -\int_{-1}^{1} u_{xx}^N u dx = \int_{-1}^{1} (u_x)^2 dx$$

But, we know that the degree of $|u^N|^\alpha u^N$ is greater than $2N - 1$. Here, we shall use the approximation of $|u^N|^\alpha u^N$ in (3). We subsitute $I_N|u^N|^\alpha u^N$ intead of $|u^N|^\alpha u^N$ where $I_N : C(\Omega) \to X_N$ is the interpolation operator.

We shall find the approximate solution $u^N \in X_N$ such that

$$\begin{aligned}
&\int_{-1}^{1} \left[u_{tt}^N(x,t) - u_{xx}^N(x,t) + I_N|u^N(x,t)|^\alpha u^N(x,t)\right] v(x)dx \\
&= \int_{-1}^{1} f(x,t)v(x)dx \qquad t > 0, \quad \text{for all} \quad v \in P_{N-2} \\
(4) \qquad &\int_{-1}^{1} \left[u^N(x,0) - u_0(x)\right] v(x)dx = 0 \\
&\int_{-1}^{1} \left[u_t^N(x,0) - u_1(x)\right] v(x)dx = 0.
\end{aligned}$$

THEOREM 1. *For some* $T > 0$,

$$\|P_{N-2}u_t^N(t)\|^2_{L_w(-1,1)} + \|u_x^N(t)\|^2 + (2/P\beta)\|u^N(t)\|^p_{L^p(-1,1)}$$
$$\leq\Big\{\|P_{N-2}u_t^N(0)\|^2_{L_w(-1,1)} + \|u_x^N(0)\|^2 + (2/P\beta\|u^N(0)\|^p_{L^p(-1,1)}$$
$$+\int_0^T \|f(s)\|^2_{L_w(-1,1)}ds\Big\}e^T$$

Proof. Take $v = P_{N-2}u_t^N$, from the left hand side first term in (4)

$$\int_{-1}^1 u_{tt}(x,t)P_{N-2}u_t^N(x,t)dx$$
$$=\int_{-1}^1 P_{N-2}u_{tt}(x,t)P_{N-2}u_t^N(x,t)(1-x^2)dx$$
$$=(1/2)\frac{d}{dt}\|P_{N-2}u_t^N(t)\|^2_{L_w(-1,1)},$$

and the second term,

$$-\int_{-1}^1 u_{xx}^N(x,t)P_{N-2}u_t^N(x,t)dx$$
$$=\int_{-1}^1 u_{xx}^N(x,t)P_{N-2}\frac{d}{dt}u^N(x,t)dx$$
$$=\int_{-1}^1 u_x^N(x,t)\frac{d}{dt}u_x^N(x,t)dx$$
$$=(1/2)\frac{2}{dt}\|u_x^N(t)\|^2.$$

Now, for $p = \alpha + 2$, refer to [4],

$$(1/p)\frac{d}{dt}\|u^N(x,t)\|^p_{L^p(-1,1)}$$
$$=\int_{-1}^1 |u^N(x,t)|^\alpha u^N(x,t)u_t^N(x,t)dx$$
$$=\int_{-1}^1 |u^N(x,t)|^\alpha u^N(x,t)(1-x^2)P_{N-2}u_t^N(x,t)dx$$

We can choose the β which satisfies $\|u^N - I_N u^N\|_w^2 = \|u^N - P_N u^N\|_w^2 + \|R_N u^N\|_w^2$

$$(1/p)\frac{d}{dt}\|u^N(x,t)\|_{L^p(-1,1)}^p$$
$$\leq \beta \int_{-1}^{1} I_N \left\{|u^N(x,t)|^\alpha u^N(x,t)\right\} P_{N-2}u_t^N(x,t)dx$$

Therefore, from the equation

$$\int_{-1}^{1} \left[u_{tt}^N(x,t) - u_{xx}^N(x,t) + I_N|u^N(x,t)|^\alpha u^N(x,t)\right] \ P_{N-2}u_t^N(x,t)dx$$
$$= \int_{-1}^{1} f(x,t) P_{N-2} u_t^N(x,t)dx$$

we obtain

$$(1/2)\frac{d}{dt}\|P_{N-2}u_t^N(t)\|_{L_w(-1,1)}^2 + (1/2)\frac{d}{dt}\|u_x^N(t)\|^2$$
$$+ (1/P\beta)\frac{d}{dt}\|u^N(t)\|_{L^P(-1,1)}^p$$
$$\leq \int_{-1}^{1} \left[u_{tt}^N(x,t) - u_{xx}^N(x,t) + I_N|u^N(x,t)|^\alpha u^N(x,t)\right] \ P_{n-2}u_t^N(x,t)dx$$
$$\leq \frac{1}{2}\|f(t)\|_{L_w(-1,1)}^2 + \frac{1}{2}\|P_{N-2}u_t^N(x)\|_{L_w(-1,1)}^2$$

$$\|P_{N-2}u_t^N(t)\|_{L_w(-1,1)}^2 + \|u_x^N(t)\|^2 + (2/P\beta)\|u^N(t)\|_{L^p(-1,1)}^p$$
$$\leq \|P_{N-2}u_t^N(0)\|_{L_w(-1,1)}^2 + \|u_x^N(0)\|^2 + (2/P\beta)\|u^N(0)\|_{L^p(-1,1)}^p$$
$$+ \int_0^t \|f(s)\|_{L_w(-1,1)}^2 ds + \int_0^t \|P_{N-2}u_t^N(s)\|^2 ds$$

Applying Gronwall's inequality we complete the proof.

This theorem shows the stability of the approximate solution of u^N for

$$0 = \int_{-1}^{1} (u^N(x,0) - u_0(x))\ u_{0_{xx}}^N dx = -\int_{-1}^{1} (u_x^N(x,0) - u_{0_x}(x))\ u_{0_x}^N dx$$

$$\int_{-1}^{1} u_x^N(x,0)u_{0_x}^N dx = \int_{-1}^{1} u_{0_x}(x)u_{0_x}^N dx$$
$$\leq c\int_{-1}^{1} u_{0_x}(x)u_{0_x}(x)dx < c\|u_0\|^2_{H_0^1(\Omega)}$$

3. Convergence

Let R_N be a projection operator from a dense subspace W of D_B upon X_N, where D_B is a set which satisfies the boundary condition of (2). For each $u \in W$, we further require $R_N u$ to satisfy the exact boundry conditions, i.e.,

$$R_N : W \to X_N \cap D_B.$$

We define the norm $\|g\|_{E^*} = \sup_{u\in E, u\neq 0} \frac{(g,u)}{\|u\|_E}$ for all $g \in E^*$ that is dual of E.

Let $e(x,t) = u^N(x,t) - R_N u$. We obtain the following theorem.

THEOREM 2. *Assume that* $|u|^\alpha u \in H^1(-1,1)$

$$\|P_{N-2}e_t(t)\|^2_{L_w(-1,1)} + \|e_x(t)\|^2$$
$$\leq \left\{\|P_{N-2}e_t(0)\|^2_{L_w(-1,1)} + \|e_x(0)\|^2 + M^2T\right\} e^T$$
$$\leq \left\{\|P_{N-2}e_t(0)\|^2_{L_w(-1,1)} + c\|e_0\|^2_{H_0^1(\Omega)} + M^2T\right\} e^T$$

Proof. From (3), we have

$$\int_{-1}^{1} \left[u_{tt}^N - u_{xx}^N + I_N|u^N(x,t)|^\alpha u^N(x,t) - f(x,t)\right] \ v(x)dx = 0$$
$$t > 0, \quad \text{for all} \quad v \in P_{N-2}$$

Take $v = e_t(x,t)$

$$0 = \int_{-1}^{1} u_{tt}^N - u_{xx}^N + I_N|u^N|^\alpha u^N - (u_{tt} - u_{xx} + |u|^\alpha u)e_t dx$$

$$= \int_{-1}^{1} (u_{tt}^N - R_N u_{tt}^N + R_N u_{tt}^N - u_{tt}) e_t dx$$
$$- \int_{-1}^{1} (u_{xx}^N - R_N u_{xx}^N + R_N u_{xx}^N - u_{xx}) e_t dx$$
$$+ \int_{-1}^{1} (I_N |u^N|^\alpha u^N - |u|^\alpha u) e_t dx$$

We get

(5)

$$\frac{1}{2}\frac{d}{dt}\|P_{N-2}e_t(t)\|^2_{L_w(-1,1)} + \frac{1}{2}\frac{d}{dt}\|e_x(t)\|^2$$
$$= \int_{-1}^{1} (u_{tt} - R_N u_{tt}^N) e_t + (R_N u_{tt}^N - u_{xx}) e_t + (|u|^\alpha u - I_N |u^N|^\alpha u^N) e_t dx$$
$$= \|P_{N-2}(u_{tt} - R_N u_{tt}^N)\| \|P_{N-2} e_t\|_{L_w(-1,1)}$$
$$+ \|P_{N-2}(R_N u_{xx}^N - u_{xx})\| \|P_{N-2} e_t\|_{L_w(-1,1)}$$
$$+ \||u|^\alpha u - I_N |u^N|^\alpha u^N\| \; \|P_{N-2} e_t\|_{L_w(-1,1)}$$

We refer to [1]: For each $v \in H_0^1(-1,1)$

$$(P_{N-2}(u_{tt} - R_N u_{tt}), v)$$
$$=(u_{tt} - R_N u_{tt}, v) - (u_{tt} - R_N u_{tt},\ v - P_{N-2} v)$$
$$=((u_{tt} - R_N u_{tt})_x,\ (\emptyset - R_N \emptyset)_x) - (u_{tt} - R_N u_{tt},\ v - P_{N-2} v)$$

where $\emptyset$ is the only function in $H_0^1(-1,1)$ satisfying $-\emptyset_{xx} = v$ then we obtain

$$\|P_{N-2}(u_{tt} - R_N u_{tt}\|_{E^*} \le C_N^{1-m} \|u_{tt}\|_{H^{m-2}(-1,1)}.$$

For each $v \in H_0^1(-1,1)$

$$(P_{N-2}(u - R_n u)_{xx}, v)$$
$$= -((u - R_n u)_x, v_x) - ((u - R_n u)_{xx}, v - p_{N-2} v)$$
$$=((u - R_n u)_x, v_x) - (u_{xx} - P_{n-2} u_{xx}, v - P_{n-2} v)$$

here we have used the fact that both $P_{N-2}u_{xx}$ and $(R_n u)_{xx}$ are orthogonal to $v - P_{N-2}v$. Using the same approximation results as before, we deduce

$$\|P_{N-2}(u - R_N u)_{xx}\|_{E^*} \leq CN^{1-m}\|u\|_{H^m(-1,1)}.$$

In Legendre approximations, for all $u \in H^m(-1,1)$

$$\begin{aligned}&\|u - I_N u\|_{H^l(-1,1)}\\ \leq& CN^{2l+\frac{1}{2}-m}\|u\|_{H^m(-1,1)} \text{ for } 0 \leq l \leq m \text{ with } m > \frac{1}{2}.\end{aligned}$$

Assume that $|u|^\alpha u \in H^1(-1,1)$ let $l = 0$. We get

$$\begin{aligned}&\||u|^\alpha u - I_N|u^N|^\alpha u^N\|_{L^2(-1,1)}\\ =&\||u|^\alpha u - I_n|u|^\alpha\|_{L^2(-1,1)} \leq CN^{\frac{1}{2}-m}\||u|^\alpha u\|_{H^m(-1,1)}.\end{aligned}$$

We may assume that $m \geq 2$.

Let $M = CN^{1-m}\|u_{tt}\|_{H^{m-2}(-1,1)} + CN^{1-m}\|u\|_{H^m(-1,1)} + CN^{-\frac{1}{2}}\||u|^\alpha u\|_{H^{m-2}(-1,1)}$ clearly $M \to 0$ as $N \to \infty$. From (5)

$$\begin{aligned}&(1/2)\frac{d}{dt}\|p_{N-2}e_t(t)\|^2_{L_w(-1,1)} + (1/2)\frac{d}{dt}\|e_x(t)\|^2\\ \leq&(1/2M^2 + (1/2)\|P_{N-2}e_t(t)\|^2_{L_w(-1,1)}\end{aligned}$$

$$\begin{aligned}&\|P_{N-2}e_t(t)\|^2_{L_w(-1,1)} + \|e_x(t)\|^2\\ \leq&\|P_{N-2}e_t(0)\|^2_{L_w(-1,1)} + \|e_x(0)\|^2\\ &+ \int_0^t M^2 ds + \int \|P_{N-2}e_t(x,s)\|^2_{L_w(-1,1)} ds\end{aligned}$$

We know that $\|e_x(0)\|^2 \leq c\|e_0\|^2_{H_0^2(\Omega)}$ and applying Gronwall's inequality we conclude the proof.

4. Numerical results

Set $u^N(x,t) = \sum_{i=0}^{N} a_i(t)l_i(x)$ is a N-degree Lagrange polynomial with $N+1$ nodes as $-1 = x_0 < x_1 < x_2 \cdots < x_n = 1$. We substitute $u^N(x,t)$ into (2), we get

$$\frac{d^2 a_i(t)}{dt^2} - l_1''(x_i)a_1(t) + \cdots + l_N''(x_i)a_n(t) + |a_i^3(t)|^\alpha a_i(t) = f(x_i,t)$$
$$i = 0,1,2,\cdots,N$$

Applying the boundry condition and the difference equation with

$$\frac{d^2 a_i(t_j)}{dt^2} = \frac{a_i(t_{j+1}) - 2a_i(t_j) + a_i(t_{j-1})}{h^2}$$
$$a_i(t_0) = 0$$
$$a_i(t_1) = hu_1(x_i)$$

where h is a mesh size and $t_j = jh$, we obtain a system of $N-1$ equations. For one example, let

$$\begin{aligned} f(x,t) =& -2\sin(\pi x) + |(t-t^2)\sin(\pi x)|^\alpha (t-t^2)\sin(\pi x) \\ &+ \pi^2(t-t^2)\sin(\pi x) \\ \frac{\partial u}{\partial t}&(x,0) = \sin(\pi x), \end{aligned}$$

then we obtain the numerical solution as table 1.

Practically, this example has exact solution such that $(t-t^2)\sin(\pi x)$. We can calculate errors. These errors are approximately less than $C\left(\frac{1}{N}\right)^{(N-1)} hN(N-1)\left(\frac{N}{2}\right)^{(N-1)} / \left(\frac{N}{2}\right)!$ in that $C\left(\frac{1}{N}\right)^{(N-1)}$ is estimated from M which is in theorem 2., and $N(N-1)\left(\frac{N}{2}\right)^{(N-1)} / \left(\frac{N}{2}\right)!$ is calculated by a second order defferentiation of N-degree Laglange polynomial. Briefly, the errors are less than $\left(\frac{1}{2}\right)^{(N-1)} hN(N-1)/\left(\frac{N}{2}\right)!$ and are independent of α.

$$K = \left(\frac{1}{2}\right)^{(N-1)} hN(N-1)/\left(\frac{N}{2}\right)! \qquad h = 0.001$$

	numerical	exact	error	k
$N = 8$	$9.91306E-3$	$9.9E-3$	$1.3096E-5$	$1.8229E-5$
$N = 12$	$9.900053E-3$	$9.9E-3$	$5.3E-8$	$8.9518E-8$

table 1. The numerical estimation of $u(1/2, 0.01)$, error and error bound. For time value t we get results by 10th iteration.

References

1. Claudio Canuto M., *Yousuff Hussaini Alfio quarteroni thomas A. Zang*, Spectral Methods in Fluid Dynamics, Springer-Verlag, 1988.
2. Dendy, J. E., *An analysis of some Galerkin schemes for the solution of ninlinear time-dependent problems*, SIAM J. Numer. Anal..
3. Dupont, T., *L estimates for Galerkin methods for second-order hyperbolic equations*, SIAM J. Numer. Anal..
4. Masanori Hosoya and Yoshio Yamada, *On some nonlinear wave equations I: local existence and regularity of solutions*, J. Fac. Sci. univ. Tokyo Sect. IA, Math. **38** (1991), 225-238.
5. Perring, J. K. and Skyrme, T. R. H, *A model unified field equation*, Nucl. Phys **31** (1962), 225-238.
6. Yves Truginy, *Product approximation for nonlinear Klein-Gordon equations*, IMA journal of Numerical Analysis **9** (1990), 449-462.

Department of Mathematics
Hoseo University
Asan 337-850, Korea