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ON THE BROWDER–HARTMAN–STAMPACCHIA VARIATIONAL INEQUALITY

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1. Introduction and Preliminaries

The Hartman-Stampacchia variational inequality was first suggested and studied by Hartman and Stampacchia [8] in finite dimensional spaces during the time establishing the base of variational inequality theory in 1960s [4]. Then it was generalized by Lions et al. [6], [9], [10], Browder [3] and others to the case of infinite dimensional spaces and was called the Browder-Hartman-Stampacchia variational inequality [3], [9], [10], and the results concerning this variational inequality have been applied to many important problems, i.e., mechanics, control theory, game theory, differential equations, optimizations, mathematical economics [1], [2], [6], [9], [10]. Recently, the Browder-Hartman-Stampaccnia variational inequality was extended to the case of set-valued monotone mappings in reflexive Banach sapces by Shih-Tan [11] and Chang [5], and under different conditions, they proved some existence theorems of solutions of this variational inequality.

The purpose of this paper is, under more weaker hypotheses and in a more general setting, to study the existence problem of solutions of the Browder-Hartman-Stampacchia variational inequality for set-valued mappings. The results presented in this paper generalize and improve some important results in [5] and [11].

Throughout this paper, Φ denotes either the real or the complex field. For a nonempty set X, 2^X will denote the family of all nonempty

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subsets of X. Let E and F be vector spaces over Φ, $< \cdot, \cdot >: F \times E \to \Phi$ be a bilinear functional. For each $x_0 \in E$ and $\epsilon > 0$, let

$$W(x_0, \epsilon) = \{y \in F : | < y, x_0 > | < \epsilon\}.$$

Let $\sigma(F, E)$ be the topology on F generated by the family $\{W(x, \epsilon) : x \in E, \ \epsilon > 0\}$ as a subbase for the neignborhood system at 0. It is easy to prove that the space F equipped with the topology $\sigma(F, E)$ is a locally convex topological vector space. Similarly, we can define the topology $\sigma(F, E)$ on E. Let E be a topological vector space. Then a subset X of E is said to be $\sigma(E, F)$-compact if X is compact with respect to $\sigma(E, F)$-topology.

Let X, Y be topological spaces and $T : X \to 2^Y$ be a set-valued mapping. The set $\{(x, y) \in X \times Y : y \in T(x)\}$ is called the graph of T, which is denoted by graph (T). If the graph (T) is a closed subset of $X \times Y$, then we say that the mapping T has a closed graph.

Let E, F be vector spaces and X be a nonempty subset of E. A mapping $T : X \to 2^F$ is said to be monotone with respect to the bilinear functional $< \cdot, \cdot >: F \times E \to \Phi$ if for any $x, y \in X, u \in T(x)$ and $w \in T(y)$, $\text{Re} < w - u, y - x > \geq 0$.

Let X, Y be subsets of E, F, respectively. A functional $\varphi : X \to (-\infty, +\infty]$ is said to be quasi-convex (resp., quasi-concave) if for any $\lambda \in (-\infty, +\infty]$, the set $\{x \in X : \varphi(x) \leq \lambda\}$ (resp., $\{x \in X : \varphi(x) \geq \lambda\}$ is convex. A functional $\varphi : X \times X \to (-\infty, +\infty]$ is said to be diagonally quasi-convex (resp., diagonally quasi-concave) in y if for any finite subset $\{y_1, y_2, \cdots, y_n\}$ of X and $y_0 \in \text{co}\{y_1, y_2, \cdots, y_n\}$, we have

$$\varphi(y_0, y_0) \leq \max_{1 \leq i \leq n} \varphi(y_0, y_i) \quad (\text{resp., } \varphi(y_0, y_0) \geq \min_{1 \leq i \leq n} \varphi(y_0, y_i)).$$

Let $\gamma \in (-\infty, +\infty]$ be a given number. A functional $\varphi : X \times X \to (-\infty, +\infty]$ is said to be γ-diagonally quasi-convex (resp., γ-diagonally quasi-concave) in y if for any finite subset $\{y_1, y_2, \cdots, y_n\}$ of X and $y_0 \in \text{co}\{y_1, \cdots, y_n\}$, we have

$$\gamma \leq \max_{1 \leq i \leq n} \varphi(y_0, y_i) \quad (\text{resp.}, \gamma \geq \min_{1 \leq i \leq n} \varphi(y_0, y_i)).$$

It is easy to see that $\varphi : X \times X \to (-\infty, +\infty]$ is convex (resp., concave) in $y \Rightarrow \varphi$ is quasi-convex (resp., quasi-concave) in $y \Rightarrow \varphi$ is diagonally quasi-convex (resp., diagonally quasi-concave) in $y \Rightarrow$ for some $\gamma \in (-\infty, +\infty], \varphi$ is γ-diagonally quasi-convex (resp., γ-diagonally quasi-concave) in y. But the converses do not hold.

Let $\varphi : X \times Y \to (-\infty, +\infty]$ and $\gamma \in (-\infty, +\infty]$. A functional φ is said to be γ-generalized quasi-convex (resp., γ-generalized quasi-concave) in y if for any $\{y_1, y_2, \cdots, y_n\} \subset Y$, there exists a finite set $\{x_1, x_2, \cdots, x_n\} \subset X$ such that for any subset $\{x_{i_1}, \cdots, x_{i_k}\} \subset \{x_1, \cdots, x_n\}$ and $x_0 \in \text{co}\{x_{i_1}, \cdots, x_{i_k}\}$,

$$\gamma \leq \max_{1 \leq j \leq k} \varphi(x_0, y_{i_j}) \quad (\text{resp., } \gamma \geq \min_{1 \leq j \leq k} \varphi(x_0, y_{i_j})).$$

It is obvious that if $E = F, X = Y$ and $\varphi : X \times X \to (-\infty, +\infty]$ is γ-diagonally quasi-convex (resp., γ-diagonally quasi-concave) in y, then φ is γ-generalized quasi-convex (resp., γ-generalized quasi-concave) in y.

Let X be a nonempty subset of E. A set-valued mapping $T : X \to 2^E$ is called a KKM mapping if for any finite subset $\{x_1, \cdots, x_n\}$ of X, $\text{co}\{x_1, \cdots, x_n\} \subset \bigcup_{i=1} T(x_i)$.

LEMMA 1. ([9]) *Let E be a Hausdorff topological vector space, X be a nonempty convex subset of E and $T : X \to 2^E$ be a KKM mapping with nonempty closed values. If there exists a $x_0 \in X$ such that $T(x_0)$ is a compact set in E, then*

$$\bigcap_{x \in X} T(x) \neq \emptyset.$$

LEMMA 2. ([2]) *Let E, F be Hausdorff topological vector spaces and X, Y be two nonempty convex subsets of E, F, respectively. Suppose further that functionals $\varphi, \psi : X \times Y \longrightarrow (-\infty, +\infty]$ satisfy the following conditions:*

(a) *for each $y \in Y$, $\varphi(x, y)$ is lower semi-continuous in x,*
(b) *for some $\gamma \in (-\infty, +\infty], \psi(x, y)$ is γ-generalized quasi-concave in y,*
(c) *for all $(x, y) \in X \times Y, \varphi(x, y) \leq \psi(x, y)$,*
(d) *there exists a $y_0 \in Y$ such that $\{x \in X : \varphi(x, y_0) \leq \gamma\}$ is a compact subset in X.*

Then there exists a $\bar{x} \in X$ *such that* $\sup_{y\in Y} \varphi(\bar{x}, y) \leq \gamma$.

REMARK 1. It follows from the proof in [2] that the closedness condition of X and Y can be removed.

LEMMA 3. *Let* E *be a topological vector space over* Φ, X *be a nonempty convex subset of* E, F *be a vector space over* Φ *with* $\sigma(F, E)$-*topology and* $< \cdot, \cdot >: F \times E \longrightarrow \Phi$ *be a bilinear functional. Suppose further that*

(a) $T : X \longrightarrow 2^F$ *is upper semi-continuous on each line segment of* X,

(b) $h : X \longrightarrow R$ *is a convex functional.*

Then for each $\bar{y} \in X$, *it follows from*

$$\sup_{u\in T(x)} Re < u, \bar{y} - x > \leq h(x) - h(\bar{y}), \quad x \in X, \tag{1}$$

that

$$\inf_{w\in T(\bar{y})} Re < w, \bar{y} - x > \leq h(x) - h(\bar{y}), \quad x \in X. \tag{2}$$

Proof. For each $x \in X$ and for any $t \in [0, 1]$, let $x_t = t_x + (1 - t)\bar{y} = \bar{y} - t(\bar{y} - x)$. Since X is convex, $x_t \in X$. Hence for all $t \in [0, 1]$, we have

$$\sup_{u\in T(x_t)} \text{Re} < u, \bar{y} - x_t) \leq h(x_t) - h(\bar{y})$$

and so

$$\begin{aligned} t\Big[\sup_{u\in T(x_t)} \text{Re} < u, \bar{y} - x >\Big] &\leq h(tx + (1 - t)\bar{y}) - h(\bar{y}) \\ &\leq th(x) + (1 - t)h(\bar{y}) - h(\bar{y}) \\ &= t[h(x) - h(\bar{y})]. \end{aligned}$$

Consequently, we have

$$\sup_{u\in T(x_t)} \text{Re} < u, \bar{y} - x > \leq h(x) - h(\bar{y}), \quad t \in [0, 1]. \tag{3}$$

For any $f \in T(\bar{y})$ and for any $\epsilon > 0$, letting

$$U(f) = \{w \in F : | < w - f, \bar{y} - x > | < \epsilon\},$$

then it follows that $U(f)$ is a $\sigma(F, E)$-open neighborhood of f and hence $G = \bigcup_{f \in T(\bar{y})} U(f)$ is a $\sigma(F, E)$-open neighborhood of $T(\bar{y})$. Since T is upper semi-continuous on line segment $L = \{x_t : t \in [0, 1]\}$, for the set G, there exists an open neighborhood N of $\bar{y}$ in L such that $T(y) \subset G$ for all $y \in N$. Besides, since $x_t \to \bar{y}$ as $t \to 0^+$, there exists a $\delta \in (0, 1)$ such that $x_t \in N$ for all $t \in (0, \delta)$ and so $T(x_t) \subset G$. Taking $t_0 \in (0, \delta)$ with $u_0 \in T(x_{t_0}) \subset G$, then there exists a $f_0 \in T(\bar{y})$ such that $u_0 \in U(f_0)$. Therefore, we have

$$| < u_0 - f_0, \bar{y} - x > | < \epsilon$$

and so $|\mathrm{Re} < f_0 - u_0, \bar{y} - x > | < \epsilon$. Combining (3) and this inequality, we have

$$\mathrm{Re} < f_0, \bar{y} - x >< \mathrm{Re} < u_0, \bar{y} - x > + \epsilon \leq h(x) - h(\bar{y}) + \epsilon.$$

This implies that

$$\inf_{w \in T(\bar{y})} \mathrm{Re} < w, \bar{y} - x > \leq h(x) - h(\bar{y}) + \epsilon.$$

By the arbitrariness of $\epsilon > 0$ and $x \in X$, we have

$$\inf_{w \in T(\bar{y})} \mathrm{Re} < w, \bar{y} - x > \leq h(x) - h(\bar{y}), \ x \in X.$$

This completes the proof.

REMARK 2. Lemma 3 extends and improves Lemma 2.5.3 in [1].

2. The Main Results

THEOREM 1. *Let E be a locally convex Hausdorff topological vector space over Φ, X be a nonempty convex subset of E, F be a vector space* *over Φ with $\sigma(F,E)$-topology and $< \cdot,\cdot >: F \times E \longrightarrow \Phi$ be a bilinear functional. Suppose further that*

(a) *$T : X \longrightarrow 2^F$ is monotone with compact values and is upper semi-continuous on each line segement of X,*

(b) *$h : X \longrightarrow R$ is lower semi-continuous in the $\sigma(E,F)$-topology and convex functional,*

(c) *there exist a $\sigma(E,F)$-compact set K and a $y_0 \in X$ such that for any $x \in X \backslash K$,*

$$\sup_{u\in T(y_0)} Re < u, x - y_0 >> h(y_0) - h(x).$$

Then there exists a $\bar{x} \in X$ such that

$$\begin{aligned}
&\sup_{y\in X}\Big[\sup_{u\in T(y)} Re < u, \bar{x} - y > +h(\bar{x}) - h(y)\Big] \\
&\quad \le \sup_{y\in X}\Big[\inf_{w\in T(\bar{x})} Re < w, \bar{x} - y > +h(\bar{x}) - h(y)\Big] \\
&\quad \le 0.
\end{aligned}$$

Proof. For $x, y \in X$, let

$$\begin{aligned}
\varphi(x,y) &= \sup_{u\in T(y)} \mathrm{Re} < u, x - y > +h(x) - h(y), \\
\psi(x,y) &= \inf_{w\in T(x)} \mathrm{Re} < w, x - y > +h(x) - h(y), \\
G(y) &= \{x \in X : \varphi(x,y) \le 0\}, \\
F(y) &= \{x \in X : \psi(x,y) \le 0\}.
\end{aligned}$$

(I) First, we verify that $\varphi, \psi : X \times X \longrightarrow (-\infty, +\infty]$ satisfy the conditions in Lemma 2.

In fact, since T is monotone, for all $x, y \in X$, $w \in T(x)$ and $u \in T(y)$, we have

$$\mathrm{Re} < w, x - y >\le \mathrm{Re} < u, x - y >$$

and hence we have

$$\begin{aligned}
&\inf_{w\in T(x)} \mathrm{Re} < w, x - y > +h(x) - h(y) \\
&\quad \ge \sup_{u\in T(y)} \mathrm{Re} < u, x - y > +h(x) - h(y),
\end{aligned}$$

i.e., $\varphi(x,y) \leq \psi(x,y)$ for all $x, y \in X$. This implies that for any $y \in X$, $F(y) \subset G(y)$. Hence we have

$$\bigcap_{y \in X} F(y) \subset \bigcap_{y \in X} G(y). \tag{4}$$

Next we prove that for each $u \in F$, $x \mapsto < u, x >$ is continuous in the $\sigma(E,F)$-topology. In fact, for any $x_0 \in E$ and $\epsilon > 0, W(u,\epsilon) = \{x \in E : | < u, x > | < \epsilon\}$ is an open neighborhood of 0 and hence the set

$$N(x_0) = x_0 + W(u,\epsilon) = \{x \in E : | < u, x - x_0 > | < \epsilon\}$$

is an open neighborhood of x_0. For any $x \in N(x_0)$, we have

$$| < u, x > - < u, x_0 > | = | < u, x - x_0 > | < \epsilon.$$

This shows that the function $x \mapsto \mathrm{Re} < u, x - y >$ is continuous on X in the $\sigma(E,F)$-topology. By the assumption, h is lower semi-continuous on X in $\sigma(E,F)$-topology, and by Proposition 1.4.6 [4], we know that the function

$$x \mapsto \varphi(x,y) = \sup_{u \in T(y)} \mathrm{Re} < u, x - y > + h(x) - h(y)$$

is lower semi-continuous in the topology $\sigma(E,F)$. Thus for each $y \in X, G(y) = \{x \in X : \varphi(x,y) \leq 0\}$ is a $\sigma(E,F)$-closed set in X. Since $\psi(x,y)$ is concave in y, $\psi(x,y)$ is 0-diagonally quasi-concave in y ($\gamma = \sup_{y \in X} \psi(x,y) = 0$). Hence $\psi(x,y)$ is 0-generalized quasi-concave in y.

Next, by the condition (c), for each $x \in X \setminus K$, we have

$$\sup_{u \in T(y_0)} \mathrm{Re} < u, x - y_0 >> h(y_0) - h(x)$$

and so $x \notin G(y_0) = \{x \in X : \varphi(x, y_0) \leq 0\}$, which means that $G(y_0) \subset K$. Since K is $\sigma(E,F)$-compact and $G(y_0)$ is $\sigma(E,F)$-closed, $G(y_0)$ is also $\sigma(E,F)$-compact.

Summing up the above discussion, from Lemma 2, there exists a $\bar{x} \in X$ such that

$$\sup_{y \in X} \varphi(\bar{x}, y) \leq 0, \quad \text{i.e.,} \bigcap_{y \in X} G(y) = \emptyset.$$

(II) Next we verify $\bigcap_{y \in X} G(y) = \bigcap_{y \in X} F(y)$.
In view of (4), it is sufficient to prove $\bigcap_{y \in X} G(y) \subset \bigcap_{y \in X} F(y)$. Suppose that this is not the case. Then there exists a $x_0 \in X$ such that a $x_0 \in \bigcap_{y \in X} G(y)$ but $x_0 \notin \bigcap_{y \in X} F(y)$. For any $y \in X$, let $x_t = (1-t)x_0 + ty \in X, t \in [0,1]$. Since we have

$$\begin{aligned} 0 &\geq \sup_{u \in T(x_t)} \operatorname{Re} < u, x_0 - x_t > + h(x_0) - h(x_t) \\ &\geq \Big[\sup_{u \in T(x_t)} \operatorname{Re} < u, x_0 - y > + h(x_0) - h(y) \Big], \end{aligned}$$

it follows that

$$\sup_{u \in T(x_t)} \operatorname{Re} < u, x_0 - y > + h(x_0) - h(y) \leq 0. \tag{5}$$

On the other hand, since $x_0 \notin \bigcap_{y \in X} F(y)$, there exists a $\bar{y} \in X$ such that $x_0 \in F(y)$, and so we have $\psi(x_0, \bar{y}) > 0$, i.e.,

$$\inf_{w \in T(x_0)} \operatorname{Re} < w, x_0 - \bar{y} > + h(x_0) - h(\bar{y}) > 0. \tag{6}$$

Letting $x_t = (1-t)x_0 + t\bar{y} \in X, t \in [0,1]$, we have

$$\begin{aligned} &\inf_{w \in T(x_t)} \operatorname{Re} < w, \bar{x}_t - \bar{y} > + h(\bar{x}_t) - h(\bar{y}) \\ &\qquad \leq (1-t) \Big[\inf_{w \in T(\bar{x}_t)} \operatorname{Re} < w, x_0 - \bar{y} > + h(x_0) - h(\bar{y}) \Big]. \end{aligned}$$

Since T is compact-valued and upper semi-continuous on each line segment of X,

$$\inf_{w \in T(\bar{x}_t)} \operatorname{Re} < w, x_0 - \bar{y} >$$

is lower semi-continuous. By (6), there exists a $\delta \in (0,1)$ such that

$$\inf_{w \in T(\bar{x}_t)} \operatorname{Re} < w, x_0 - \bar{y} > + h(x_0) - h(\bar{y}) > 0, \quad t \in (0, \delta),$$

which contradicts (5). The assertion (II) is proved.

(III) Combining (I) and (II), we know that

$$\bigcap_{y \in X} G(y) = \bigcap_{y \in X} F(y) \neq \emptyset.$$

Taking $\bar{x} \in \bigcap_{y \in X} G(y) = \bigcap_{y \in X} F(y)$, we have

$$\inf_{u \in T(y)} \text{Re} < u, \bar{x} - y > +h(\bar{x}) - h(y) \leq 0, \quad y \in X,$$
$$\inf_{w \in T(x)} \text{Re} < w, \bar{x} - y > +h(\bar{x}) - h(y) \leq 0, \quad y \in X.$$

Therefore, noting $\varphi(x, y) \leq \psi(x, y)$ for all $x, y \in X$, we have

$$\begin{aligned} &\sup_{y \in X} \Big[\sup_{u \in T(y)} \text{Re} < u, \bar{x} - y > +h(\bar{x}) - h(y) \Big] \\ (7) \qquad &\leq \sup_{y \in X} \Big[\inf_{w \in T(\bar{x})} \text{Re} < w, \bar{x} - y > +h(\bar{x}) - h(y) \Big] \\ &\leq 0. \end{aligned}$$

This completes the proof.

COROLLARY 1. *Under the conditions of Theorem 1, in addition, if $T(\bar{x})$ is convex, then there exists a $\bar{w} \in T(x)$ such that*

$$Re < \bar{w}, \bar{x} - y > \leq h(y) - h(\bar{x}), \quad y \in X.$$

Proof. First we prove that the function $f \mapsto \text{Re} < f, x >$ is continuous on F in $\sigma(F, E)$-topology. In fact, for any $f_0 \in F$ and for any $\epsilon > 0$, since

$$W(x, \epsilon) = \{y \in F : | < y, x > | < \epsilon\}$$

is an open neighborhood of 0 in F, the set

$$V = f_0 + W(x, \epsilon) = \{f \in F : | < f - f_0, x > | < \epsilon\}$$

is an open neighborhood of f_0. For $f \in V$, we have

$$\begin{aligned} |\text{Re} < f, x > - \text{Re} < f_0, x > | &\leq |\text{Re} < f - f_0, x > | \\ &\leq | < f - f_0, x > | < \epsilon. \end{aligned}$$

By the arbitrariness of f_0, the assertion is proved.

Now we define a function $\varphi : X \times T(x) \longrightarrow R$ as follows:

$$\varphi(y,w) = \text{Re} < w, \bar{x} - y > + h(\bar{x}) - h(y), \ (y,w) \in X \times T(x).$$

Then for any $y \in X$, $w \mapsto \varphi(y,w)$ is a continuous affine function in $\sigma(F,E)$-topology on $T(\bar{x})$, and for each $w \in T(\bar{x})$, the function $y \mapsto \varphi(y,w)$ is concave on X. Since X is nonempty convex and $T(\bar{x})$ is $\sigma(F,E)$-compact convex subset, by Ky Fan's minimax theorem (see, for example, Theorem 3.7.4 [4]), we have

$$\min_{w \in T(x)} \sup_{y \in X} \varphi(y,w) = \sup_{y \in X} \min_{w \in T(\bar{x})} \varphi(y,w).$$

It follows from (7) that

$$\min_{w \in T(\bar{x})} \sup_{y \in X} [\text{Re} < w, \bar{x} - y > + h(\bar{x}) - h(y)] \leq 0.$$

Since $T(\bar{x})$ is compact, there exists a $\bar{w} \in T(\bar{x})$ such that

$$\begin{aligned} &\sup_{y \in X} [\text{Re} < \bar{w}, \bar{x} - y > + h(\bar{x}) - h(y)] \\ &\quad = \min_{w \in T(\bar{x})} \sup_{y \in X} [\text{Re} < w, \bar{x} - y > + h(\bar{x}) - h(y)], \end{aligned}$$

which shows that

$$\text{Re} < \bar{w}, \bar{x} - y > \leq h(y) - h(\bar{x}), \quad y \in X.$$

This completes the proof.

REMARK 3. When E is a reflexive Banach space, $F = E^*$ and $< \cdot, \cdot >$ is the pairing between E and E^*, it is obvious that the $\sigma(F,E)$-topology on F is just the weak topology on E^* (since E is reflexive, the weak topology on E just coincides with the weak* topology on E). Taking $h \equiv 0$, by the condition that there exists a $y_0 \in X$ such that

$$\lim_{\|x\| \to \infty} \sup_{u \in T(y_0)} \text{Re} < u, x - y_0 > > 0, \quad x \in X$$

there exists a number $r > 0$ such that for any $x \in X$ with $\|x\| > r$, we have

$$\sup_{u \in T(y_0)} \operatorname{Re} < u, x - y_0 >> 0. \tag{8}$$

Let $K = \{x \in X : \|x\| \leq r\}$, then K is a weak compact subset of X. For any $x \in X \backslash K$, it satisfies (8). Hence by Theorem 1 and Corollary 1, we can obtain the conclusion of Theorem 4.1 [5] as a special case. Moreover, in Theorem 1 and Corollary 1, X does not require to be closed and hence Theorem 1 and Corollary 1 also improve the results of [2].

THEOREM 2. *Let E be a locally convex Hausdorff topological vector space, X be a nonempty convex subset of E, F be a vector space over Φ with $\sigma(F, E)$-topology and $< \cdot, \cdot >: F \times E \longrightarrow \Phi$ be a bilinear functional. Suppose further that*

(a) *$T : X \longrightarrow 2^F$ is monotone and upper semi-continuous on each line segement of X,*
(b) *$h : X \longrightarrow R$ is convex and lower semi-continuous in the $\sigma(E, F)$-topology,*
(c) *there exist a $\sigma(E, F)$-compact subset K and a $x_0 \in X$ such that for any $y \in X \backslash K$*

$$\inf_{w \in T(y)} Re < w, y - x_0 >> h(x_0) - h(y).$$

Then there exists a $y \in X$ such that

$$\inf_{w \in T(\bar{y})} Re < w, \bar{y} - x > \leq h(x) - h(\bar{y}), \quad x \in X.$$

Proof. For each $x \in X$, let

$$F(x) = \big\{y \in X : \inf_{w \in T(y)} \operatorname{Re} < w, y - x > +h(y) - h(x) \leq 0\big\},$$

$$G(x) = \big\{y \in X : \sup_{u \in T(x)} \operatorname{Re} < u, y - x > +h(y) - h(x) \leq 0\big\}.$$

(I) First we prove that $\bigcap_{x \in X} F(x) = \bigcap_{x \in X} G(x)$.

In fact, since T is monotone, for any $x, y \in X$, we have

$$\begin{aligned}\inf_{w\in T(y)} \operatorname{Re} < w, y-x > +h(y)-h(x) \\ \geq \sup_{u\in T(x)} \operatorname{Re} < u, y-x > +h(y)-h(x),\end{aligned}$$

which implies that for each $x \in X$,

(9) $$F(x) \subset G(x).$$

Hence we have

(10) $$\bigcap_{x\in X} F(x) \subset \bigcap_{x\in X} G(x)$$

On the other hand, by using Lemma 3, if $y \in X$ such that

$$\sup_{u\in T(x)} \operatorname{Re} < u, y-x > +h(y)-h(x) \leq 0, \quad x \in X,$$

then we can deduce that

$$\inf_{w\in T(y)} \operatorname{Re} < w, y-x > +h(y)-h(x) \leq 0, \quad x \in X.$$

Thus for each $x \in X$, we have

(11) $$G(x) \subset F(x).$$

This implies that $\bigcap_{x\in X} G(x) \subset \bigcap_{x\in X} F(x)$. Combining (10) and (11), the assertion is proved.

(II) Next we prove that $G : X \longrightarrow 2^X$ is a KKM mapping.

Suppose that the mapping G is not a KKM mapping. Then there exist a finite set $\{x_1, \cdots, x_n\} \subset X$ and a $\bar{y} \in \operatorname{co}\{x_1, \cdots, x_n\}$, $\bar{y} = \sum_{i=1}^n \lambda_i x_i$, where $\lambda_i \geq 0, i = 1, 2, \cdots, n$, and $\sum_{i=1}^n \lambda_i = 1$ such that $\bar{y} \notin \bigcup_{i=1}^n G(x_i)$. By (9), we have $\bar{y} \notin \bigcup_{i=1}^n F(x_i)$, which implies that

$$\inf_{w\in T(\bar{y})} \operatorname{Re} < w, \bar{y}-x_i > +h(y)-h(x_i) > 0, \quad i = 1, 2, \cdots, n.$$

On the other hand, since

$$\begin{aligned}
0 &= \inf_{w \in T(\bar{y})} \operatorname{Re} < w, \bar{y} - \bar{y} > + h(\bar{y}) - h(\bar{y}) \\
&= \inf_{w \in T(\bar{y})} \operatorname{Re} < w, \bar{y} - \sum_{i=1}^{n} \lambda_i x_i > + h(\bar{y}) - h(\sum_{i=1}^{n} \lambda_i x_i) \\
&\geq \sum_{i=1}^{n} \lambda_i [\operatorname{Re} < w, \bar{y} - x_i > + h(\bar{y}) - h(x_i)] \\
&> 0,
\end{aligned}$$

which is a contradiction. Therefore, G is a KKM mapping.

(III) Now we prove that $\bigcap_{x \in X} F(x) = \bigcap_{x \in X} G(x) \neq \emptyset$.

In fact, by the condition (c), there exist a $\sigma(E, F)$-compact set K and a $x_0 \in X$ such that for any $y \in X \backslash K$

$$\inf_{w \in T(y)} \operatorname{Re} < w, y - x_0 > + h(y) - h(x_0) > 0.$$

This means that $y \notin F(x_0)$ and so $F(x_0) \subset K$. By (11), we have

$$G(x_0) \subset K. \tag{12}$$

Besides, in Theorem 1 we have proved that the function $y \mapsto < u, y >$ is continuous in the $\sigma(E, F)$-topology and hence for each $u \in F$ and $x \in X$, the function $y \mapsto \operatorname{Re} < u, y - x >$ is continuous in $\sigma(E, F)$-topology in X. By Proposition 1.4.6 [4], we know that the function $y \mapsto \sup_{u \in T(x)} \operatorname{Re} < u, y - x >$ is lower semi-continuous in $\sigma(E, F)$-topology on X. By assumption h is lower semi-continuous in the $\sigma(E, F)$-topology on X. Therefore, $G(x)$ is a $\sigma(E, F)$-closed set. By using (12) and noting that K is a $\sigma(E, F)$-compact subset, we know that $G(x_0)$ is a $\sigma(E, F)$-compact set. Thus, by Lemma 1 , we have

$$\bigcap_{x \in X} G(x) \neq \emptyset.$$

Thus, combining the conclusion in (I), the assertion is proved.

(IV) Finally we prove the conclusion of Theorem 2.

Taking $\bar{y} \in \bigcap_{x \in X} F(x)$, then we have

$$\inf_{w \in T(\bar{y})} \operatorname{Re} < w, \quad \bar{y} - x > \leq h(x) - h(\bar{y}), \quad x \in X.$$

This completes the proof.

REMARK 4. Theorem 2 improves the corresponding results of [3] in some aspects, such as (a) for each $x \in X$, $T(x)$ does not require to be a weak compact subset, (b) X needs not to be closed in E.

COROLLARY 2. *Under the conditions of Theorem 2, in addition, if $T(y)$ is a compact convex subset in F, then there exists a $\bar{w} \in T(\bar{y})$ such that*

$$Re < \bar{w}, \bar{y} - x > \leq h(x) - h(\bar{y}), \quad x \in X.$$

The proof of this corollary is similar to the proof of Corollary 1 and so it is omitted here.

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