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AN EMPIRICAL CLT FOR STATIONARY MARTINGALE DIFFERENCES

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1. Introduction and main result

Let S be a set and $\mathcal{B}$ be a σ-field on S. We consider $(\Omega = S^Z, \mathcal{T} = \mathcal{B}^Z, P)$ as the basic probability space. We denote by T the left shift on Ω. We assume that P is invariant under T, i.e., $PT^{-1} = P$, and that T is ergodic. We denote by $X = \cdots, X_{-1}, X_0, X_1, \cdots$ the coordinate maps on Ω. From our assumptions it follows that $\{X_i\}_{i\in Z}$ is a stationary and ergodic process. Next we define for each $i \in Z$ a σ-fields $M_i := \sigma(X_j : j \leq i)$ and $H_i := \{f : \Omega \to R : f \in M_i \text{ and } f \in L^2(\Omega)\}$. We denote for each $f \in L^2(\Omega)$, $E_{i-1}(f) := E(f|M_{i-1})$, and $H_0 \ominus H_{-1} := \{f \in H_0 : E(f \cdot g) = 0 \text{ for each } g \in H_{-1}\}$. Finally for every $f,\ g \in L^2(\Omega)$ we put $d(f,g) := [E(f-g)^2]^{1/2}$. We assume $\mathcal{F} \subseteq H_0 \ominus H_{-1}$. From our setup it follows that for every $f \in \mathcal{F}$, $\{f(T^i(X)), M_i\}$ is a stationary martingale difference sequence. For every $f \in \mathcal{F}$, we define

$$S_n(f) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f(V_i), \tag{1}$$

where $V_i := T^i(X)$ and $V := T^0(X)\ (= X)$.

Our goal is to find sufficient conditions for an empirical central limit theorem. This essentially means showing that $\mathcal{L}(S_n(f) : f \in \mathcal{F}) \to \mathcal{L}(G(f) : f \in \mathcal{F})$, where the processes that are involved here are indexed by $\mathcal{F}$ and are considered as random elements in $B(\mathcal{F})$, the space of the bounded real-valued functions on $\mathcal{F}$, taken with the sup norm. $(G(f) : f \in \mathcal{F})$ is a Gaussian process which is continuous in f a.s.. Next we define the metric entropy with bracketing. See, for example, Dudley (1984).

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DEFINITION 1. For $(\mathcal{F}, d)$ and $\delta > 0$ we define the covering number with bracketing $\nu^B(\delta, \mathcal{F}, d)$, or $\nu^B(\delta)$ if there is no risk of ambiguity, as the smallest n for which there exists $\{f^l_{0,\delta}, f^u_{0,\delta}, \cdots, f^l_{n,\delta}, f^u_{n,\delta}\} \subseteq H_0$ so that for every $f \in \mathcal{F}$ there exist some $0 \le i \le n$ satisfying

$$(a)\ f^l_{i,\delta} \le f \le f^u_{i,\delta}$$

and

$$(b)\ d(f^l_{i,\delta}, f^u_{i,\delta}) < \delta.$$

Define the metric entropy with bracketing to be

$$H^B(\delta) := H^B(\delta, \mathcal{F}, d) := \ln \nu^B(\delta, \mathcal{F}, d).$$

We also define the associated integral for $0 < \delta \le 1$

$$J(\delta) := \int_0^\delta [H^B(u)]^{\frac{1}{2}} du.$$

We use the following notations: For a function $\varphi : \mathcal{F} \to R$, we let $||\varphi||_{\mathcal{F}} := \sup_{f \in \mathcal{F}} |\varphi(f)|$ denote the sup of $|\varphi|$ over $\mathcal{F}$. We write $||\cdot||$ in stead of $||\cdot||_{\mathcal{F}}$ when there is no risk of ambiguity. We also let $||\varphi||_\delta := \sup_{(\delta)} |\varphi(f) - \varphi(g)|$ denote the sup of $|\varphi(f) - \varphi(g)|$ over (δ) where $(\delta) := \{(f, g) \in \mathcal{F} \times \mathcal{F} : d(f, g) < \delta\}$.

We are now ready to state our main result.

THEOREM 1. (An eventual uniform equicontinuity). *Assume that*

(a) $J(1) = \int_0^1 [H^B(u)]^{\frac{1}{2}} du < \infty$ *and*

(b) *there exists a constant $D > 0$ such that*

$$P^*\{\sup_{f, g \in H_0} \sum_{i=1}^n \frac{E_{i-1}[f(V_i) - g(V_i)]^2}{n d^2(f, g)} \ge D\} \to 0.$$

Then for every $\epsilon > 0$ there is $\delta > 0$ such that

$$\limsup_n P^*\{||S_n||_\delta \ge \epsilon\} \le \epsilon, \tag{2}$$

where P^* denotes outer probability.

In the following Corollary 1 we state an empirical central limit theorem for martingale differences. It is well known that $B(\mathcal{F})$ is complete in the sup-norm, so that $(B(\mathcal{F}), ||\cdot||_{\mathcal{F}})$ form a Banach space. We use the following definition of weak convergence due to Hoffmann-Jϕrgensen (see Hoffmann-Jϕrgensen, 1991, p 149).

DEFINITION 2. A sequence of $B(\mathcal{F})$-valued random functions $\{Y_n : n \geq 1\}$ converges in law to a $B(\mathcal{F})$-valued Borel measurable random function Y, denoted $Y_n \Rightarrow Y$, if

$$Eg(Y) = \lim_{n\to\infty} E^*g(Y_n), \forall g \in C(B(\mathcal{F}), ||\cdot||_{\mathcal{F}}),$$

where $C(B(\mathcal{F}), ||\cdot||_{\mathcal{F}})$ is the set of all bounded, continuous functions from $(B(\mathcal{F}), ||\cdot||_{\mathcal{F}})$ into R. Here E^* denotes upper expectation.

COROLLARY 1. *Under the assumptions of Theorem 1,*

$$S_n \Rightarrow G,$$

where $G(f)$ is a Gaussian process with $EG(f) = 0$ and $EG(f_1)G(f_2) = Ef_1(X)f_2(X)$ which is uniformly continuous in f a.s..

Proof of Corollary 1. The proof follows from the finite dimensional convergence and the eventual uniform equicontinuity of S_n. □

The following remarks verify that the two conditions (the finite dimensional convergence and the eventual uniform equicontinuity) are sufficient for the proof of the Corollary 1 (see Andersen (1985) and Andersen and Dobric (1987) for the similar argument of i.i.d. setup).

REMARK 1. $\{S_n\}$ is eventually bounded. I.e.
$\lim_{a\to\infty} \limsup_n P^*\{||S_n||_{\mathcal{F}} > a\} = 0$. Indeed, note that $\forall\epsilon > 0$, $\exists\delta > 0$ such that

$$\limsup_n P^*\left\{\sup_{d(f_1,f_2)\leq\delta} |S_n(f_1) - S_n(f_2)| \geq \epsilon\right\} \leq \epsilon.$$

Let A be the finite set of *the* δ-nets. Then, by the finite dimensional convergence, we have

$$\lim_{a\to\infty} \limsup_n P^*\{\sup_{\alpha\in A} |S_n(f_\alpha)| > a\} = 0.$$

We write $M_n := \sup_{\alpha\in A} |S_n(f_\alpha)|$. Then note that

$$||S_n||_{\mathcal{F}} \leq M_n + \sup_{d(f_1,f_2)\leq\delta} |S_n(f_1) - S_n(f_2)| \ a.s..$$

Then we have

$$\begin{aligned}&\lim_{a\to\infty}\limsup_n P^*\{||S_n||_{\mathcal{F}} > a+\epsilon\}\\ &\le \limsup_n P^*\{||S_n||_{\mathcal{F}} - M_n > \epsilon\}\\ &\quad + \lim_{a\to\infty}\limsup_n P^*\{M_n > a\} < \epsilon.\end{aligned}$$

Letting $\epsilon \to 0$, we get the eventual boundedness of $\{S_n\}$.

REMARK 2. $\{S_n\}$ is eventually tight. I.e. $\forall \epsilon > 0$, $\exists$ a compact set K such that $\limsup_n P^*\{S_n \notin G\} < \epsilon$ for all open sets G so that $G \supseteq K$. Indeed, the eventual uniform equicontinuity and the eventual boundedness together imply the eventual tightness of $\{S_n\}$ (see Andersen and Dobric (1987), Theorem 2.12).

REMARK 3. Apply Theorem 7.11 (case 3, Remark (1)) in Hoffmann-Jϕrgensen (1991) to conclude that $S_n \Rightarrow G$. Indeed, we consider

$$\Psi := \left\{ e^{i\sum_k a_k f_k} : a_k \in R, f_k \in \mathcal{F}, \text{ and } \sum_k a_k f_k \text{ is a finite sum} \right\}$$

where $e^{i\sum_k a_k f_k}(t) := e^{i\sum_k a_k t(f_k)}$. Then Ψ is a selfadjoint semigroup of bounded, continuous complex-valued functions on $B(\mathcal{F})$. By the finite dimensional convergence of $\{S_n\}$, we have that $\lim_n E\psi(S_n)$ exists $\forall \psi \in \Psi$. If $t_1 \neq t_2$, then we can find $\psi \in \Psi$ such that $\psi(t_1) \neq \psi(t_2)$. Also we can find $f \in \mathcal{F}$ such that $t_1(f) \neq t_2(f)$. Choose $a \neq 0$ so that $-\pi < at_1(s) < \pi$ and $-\pi < at_2(s) < \pi$. Then $e^{iaf}(t_1) = e^{iat_1(f)} \neq e^{iat_2(f)} = e^{iaf}(t_2)$. This shows that Ψ separates points in $B(\mathcal{F})$.

REMARK 4. See Theorem 4.1 in Andersen and Dobric (1987) for the uniform continuity of G.

We observe that the assumption (a) in the theorem implies the total boundedness of the metric space $(\mathcal{F}, d)$ and the assumption (b) is an asymptotic Lipschitz condition in the average sense with a Lipschitz constant D.

Theorem 1 can be considered as a generalization of Theorem 3.1 of Ossiander (1987). To specialize our work to their framework we assume that $P = (P_0)^Z$ for some P_0, a probability measure on S (in other words the X_i are i.i.d) and that all the functions in $\mathcal{F}$ depend on X_1 coordinate only and $E(f(X_1)) = 0$. In that case the condition (b) in Theorem 1 boils down to the Lipschitz condition (2.3) in that paper.

Theorem 1 can be also considered as a generalization of Theorem 2 of Levental (1989) in the following sense. We remove the uniform boundedness requirement of underlying martingale difference sequence. We note that the condition (b)(i) in that paper is weaker than our condition (b). The other two conditions (a) and (b)(ii) together are similar to our condition (a) about the integrability of metric entropy with bracketing. We also note that we use stationarity in one place in the proof of our Theorem 1 while it was not used in Levental (1989).

2. Proof of Theorem 1

For $a > 0$, let

$$\psi(a,x) = \begin{cases} a & \text{if } a < x \\ x & \text{if } -a \le x \le a \\ -a & \text{if } x < -a. \end{cases}$$

For each $\theta > 0$, $n \ge 1$ and $f \in \mathcal{F}$, let

$$f^{(\sqrt{n}\theta)}(\cdot) = \psi(\sqrt{n}\theta, f(\cdot))$$

and

$$S_n^{(\theta)}(f) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ f^{(\sqrt{n}\theta)}(V_i) - E_{i-1}(f^{(\sqrt{n}\theta)}(V_i)) \right\}.$$

PROPOSITION 1. *Assume that*

(a) $J(1) < \infty$ *and*
(b) *there exists a constant $D > 0$ such that*

$$P^* \left\{ \sup_{f,g \in H_0} \sum_{i=1}^{n} \frac{E_{i-1}[f(V_i) - g(V_i)]^2}{n d^2(f,g)} \ge D \right\} \to 0.$$

Then for every $\eta > 0$, for every $\delta > 0$, and for each

$\theta \le \frac{\delta}{2}(\frac{D}{8(2H^B(\delta)+\eta^2)})^{1/2}$

$$P^*\left\{||S_n^{(\theta)}||_\delta \ge K\sqrt{D}(J(\delta)+\eta\delta)\right\} \le 5\sum_{k=0}^{\infty}\exp\{-\eta^2 Lk\} + o(1)$$

where K is a universal constant and $Lx = \ln(x \vee e)$.

Proof of Theorem 1. Fix $\eta > 0$. The family $\{f(\cdot) : f \in \mathcal{F}\}$ is uniformly bounded by an envelope $F(\cdot) := \sup_{f\in\mathcal{F}}|f(\cdot)| \in L^2(\Omega)$ because $\nu^B(1) < \infty$ and $F(\cdot) \le \sum_{i=0}^{\nu^B(1)}(|f_{i,1}^l(\cdot)| + |f_{i,1}^u(\cdot)|) \in L^2(\Omega)$. Note that $\{f(V_i)\}$ is a martingale difference sequence. So we have

$$|E_{i-1}(f(V_i)1_{\{|f(V_i)|>\sqrt{n}\theta\}})| = |E_{i-1}(f(V_i)1_{\{|f(V_i)|\le\sqrt{n}\theta\}})|.$$

Note also that for $\theta > 0$, $f \in \mathcal{F}$, we have

$$\begin{aligned}
&\sup_{f\in\mathcal{F}}|S_n(f) - S_n^{(\theta)}(f)| \\
&= \sup_{f\in\mathcal{F}}\frac{1}{\sqrt{n}}\left|\sum_{i=1}^n\left\{f(V_i) - f^{(\sqrt{n}\theta)}(V_i) + E_{i-1}(f^{(\sqrt{n}\theta)}(V_i))\right\}\right| \\
&\le \sup_{f\in\mathcal{F}}\frac{1}{\sqrt{n}}\sum_{i=1}^n\left|f(V_i) - f^{(\sqrt{n}\theta)}(V_i)\right| + \sup_{f\in\mathcal{F}}\frac{1}{\sqrt{n}}\sum_{i=1}^n\left|E_{i-1}(f^{(\sqrt{n}\theta)}(V_i))\right| \\
&\le \frac{1}{\sqrt{n}}\sum_{i=1}^n F(V_i)1_{\{F(V_i)>\sqrt{n}\theta\}} \\
&\quad + \sup_{f\in\mathcal{F}}\frac{1}{\sqrt{n}}\sum_{i=1}^n\left|E_{i-1}(f(V_i)1_{\{|f(V_i)|\le\sqrt{n}\theta\}})\right| \\
&\quad + \sup_{f\in\mathcal{F}}\frac{1}{\sqrt{n}}\sum_{i=1}^n\left|E_{i-1}(\sqrt{n}\theta)1_{\{|f(V_i)|>\sqrt{n}\theta\}})\right| \\
&\le \frac{1}{\sqrt{n}}\sum_{i=1}^n F(V_i)1_{\{F(V_i)>\sqrt{n}\theta\}} + \sup_{f\in\mathcal{F}}\frac{2}{\sqrt{n}}\sum_{i=1}^n E_{i-1}(|f(V_i)|1_{\{|f(V_i)|>\sqrt{n}\theta\}}) \\
&\le \frac{1}{\sqrt{n}}\sum_{i=1}^n F(V_i)1_{\{F(V_i)>\sqrt{n}\theta\}} + \frac{2}{\sqrt{n}}\sum_{i=1}^n E_{i-1}(F(V_i)1_{\{F(V_i)>\sqrt{n}\theta\}}) \\
&\le \frac{1}{\theta n}\sum_{i=1}^n F^2(V_i)1_{\{F(V_i)>\sqrt{n}\theta\}} + \frac{2}{\theta n}\sum_{i=1}^n E_{i-1}F^2(V_i)1_{\{F(V_i)>\sqrt{n}\theta\}}.
\end{aligned}$$

The last two terms converges in $L^2(\Omega)$ to zero because the stationarity and Dominated Convergence Theorem together imply

$$\frac{1}{n}\sum_{i=1}^{n} E(F^2(V_i)1_{\{F(V_i)>\sqrt{n}\theta\}}) = E(F^2(V)1_{\{F(V)>\sqrt{n}\theta\}}) = o(1).$$

Therefore we have $P\{||S_n - S_n^{(\theta)}|| > \frac{\epsilon}{4}\} = o(1)$. Since $||S_n||_\delta = ||S_n^{(\theta)}||_\delta$ $+2||S_n - S_n^{(\theta)}||$, it remains to show

$$P\left\{||S_n^{(\theta)}||_\delta > \frac{\epsilon}{2}\right\} \leq \frac{\epsilon}{2}. \tag{3}$$

We may choose η so that $5\sum_{k=0}^{\infty}\exp\{-\eta^2 Lk\} \leq \frac{\epsilon}{2}$. Now choose δ small enough so that $K\sqrt{D}(J(\delta)+\eta\delta) \leq \frac{\epsilon}{2}$. Then by Proposition 1 (3) is true for $\theta \leq \frac{\delta}{2}(\frac{D}{8(2H^B(\delta)+\eta^2)})^{1/2}$ and n large enough. End of proof of Theorem 1.

Proof of Proposition 1. We define a stopping time τ_n, for $n \geq 1$

$$\tau_n := n \wedge \max\left\{k \geq 0 : \sup_{f,g\in H_0}\sum_{i=1}^{k}\frac{E_{i-1}[f(V_i)-g(V_i)]^2}{nd^2(f,g)} < D\right\}.$$

Then from (b) we get $P\{\tau_n < n\} \to 0$ as $n \to \infty$. Note that

$$P\left\{\sup_{f,g\in H_0}\sum_{i=1}^{\tau_n}\frac{E_{i-1}[f(V_i)-g(V_i)]^2}{nd^2(f,g)} \geq D\right\} = 0. \tag{4}$$

We write

$$S_{\tau_n}^{(\theta)}(f) = \frac{1}{\sqrt{n}}\sum_{i=1}^{\tau_n}\left\{f^{(\sqrt{n}\theta)}(V_i) - E_{i-1}(f^{(\sqrt{n}\theta)}(V_i))\right\}. \tag{5}$$

Since $P\{\tau_n < n\} \to 0$ as $n \to \infty$, it is enough to prove that for every $\eta > 0$, for every $\delta > 0$, and for each $\theta \leq \frac{\delta}{2}(\frac{D}{8(2H^B(\delta)+\eta^2)})^{1/2}$

$$P^*\left\{||S_{\tau_n}^{(\theta)}||_\delta \geq K\sqrt{D}(J(\delta)+\eta\delta)\right\} \leq 5\sum_{k=0}^{\infty}\exp\{-\eta^2 Lk\}$$

In order to prove the last inequality we follow the steps in Ossiander (1987).

Step 1 : Fix $\eta > 0$, and fix $\delta > 0$. For $k \geq 0$, let $\delta_k = \frac{\delta}{2^k}$ and $\gamma_k = \sum_{j=o}^{k} H^B(\delta_j)$. Let $\{a_k : k \geq 0\}$ be a strictly decreasing sequence with $\lim_{k\to\infty} a_k = 0$. The values of a_k will be specified after applying Freedman inequality below. We write $I_k := [a_{k+1}, a_k)$ and $\bar{I}_k := [a_{k+1}, \infty)$. Note that the intervals I_k is a partition of the interval $(0, a_0)$.

Step 2 : Fix $\theta \leq \frac{a_0}{2}$. We construct a *nested* sequence of upper and lower δ_k-approximations to $f^{(\sqrt{n}\theta)}$ in $L^2(\Omega)$ in the following way. For $f \in \mathcal{F}$, let

$$u_k(\cdot) = \bigwedge_{j=0}^{k} f^u_{i_j,\delta_j}(\cdot), \quad \text{and } l_k(\cdot) = \bigvee_{j=0}^{k} f^l_{i_j,\delta_j}(\cdot),$$

where i_j is the i that satisfies (a) and (b) in the Definition 1 for δ_j. Let

$$u_{n,k}(\cdot) = \psi(\sqrt{n}\theta, u_k(\cdot)), \quad \text{and } l_{n,k}(\cdot) = \psi(\sqrt{n}\theta, l_k(\cdot)).$$

Note that $l_{n,k}$ and $u_{n,k}$ depend on $\mathcal{F}$ only through $f^l_{i_0,\delta_0}, \cdots, f^l_{i_k,\delta_k}$ and $f^u_{i_0,\delta_0}, \cdots, f^u_{i_k,\delta_k}$ respectively. Observe that

$$\sup_{f\in\mathcal{F}} \left| f^{(\sqrt{n}\theta)}(\cdot) \right| \leq \frac{a_0\sqrt{n}}{2},$$

$$\sup_{f\in\mathcal{F}} |u_{n,k}(\cdot)| \vee |l_{n,k}(\cdot)| \leq \frac{a_0\sqrt{n}}{2},$$

and

$$\sup_{f\in\mathcal{F}} |u_{n,k}(\cdot) - l_{n,k}(\cdot)| \leq a_0\sqrt{n}.$$

Note that for each $k \geq 0$,

$$l_{n,k}(\cdot) \leq f^{(\sqrt{n}\theta)}(\cdot) \leq u_{n,k}(\cdot),$$

and

$$0 \leq u_{n,k+1}(\cdot) - l_{n,k+1}(\cdot) \leq u_{n,k}(\cdot) - l_{n,k}(\cdot).$$

Step 3 : We construct the sets with which we partition $S_{\tau_n}^{(\theta)}$. Choose k_n so that

$$na_{k_n+1} < (J(\delta) + \eta\delta)\sqrt{D} \leq na_{k_n}.$$

For $0 \leq k \leq k_n$, define the following subsets of the sample space

$$A_{n,k}(f) = \left[\frac{u_{n,k}(\cdot) - l_{n,k}(\cdot)}{\sqrt{n}} \in I_k\right],$$

and

$$\tilde{A}_{n,k}(f) = \left[\frac{u_{n,k}(\cdot) - l_{n,k}(\cdot)}{\sqrt{n}} \in \bar{I}_k\right].$$

The sets $\{B_{n,k}(f) : 0 \leq k \leq k_n + 1\}$ are partitions of the sample space induced by the sets $\{\tilde{A}_{n,k}(f), 0 \leq k \leq k_n\}$:

$$B_{n,0}(f) = A_{n,0}(f),$$

$$\begin{aligned} B_{n,k}(f) =& \tilde{A}_{n,k}(f) \setminus \bigcup_{j=0}^{k-1} \tilde{A}_{n,j}(f) \\ =& A_{n,k}(f) \setminus \bigcup_{j=0}^{k-1} \tilde{A}_{n,j}(f),\ 1 \leq k \leq k_n, \end{aligned}$$

and

$$B_{n,k_n+1}(f) = (\bigcup_{k=0}^{k_n} B_{n,k}(f))^c.$$

For $k \geq 1$, let

$$C_{n,k}(f) = \bigcup_{j=k}^{k_n+1} B_{n,j}(f).$$

Since $C_{n,k}(f) \subset \tilde{A}^c_{n,k-1}(f)$, we have, on the set $C_{n,k}(f)$,

$$l_{n,k}(\cdot) - l_{n,k-1}(\cdot) \leq u_{n,k-1}(\cdot) - l_{n,k-1}(\cdot) \leq a_k\sqrt{n}. \tag{6}$$

Step 4 : In this step we stratify $S_{\tau_n}^{(\theta)}(f)$ using the partition $\{B_{n,k}(f) : 0 \le k \le k_n + 1\}$ constructed in Step 3. For $0 \le k \le k_n + 1$, let

$$S_{\tau_n,k}(f) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_n} \left\{ f^{(\sqrt{n}\theta)}(V_i) 1_{B_{n,k}(f)}(V_i) - E_{i-1}(f^{(\sqrt{n}\theta)}(V_i) 1_{B_{n,k}(f)}(V_i)) \right\},$$

and

$$L_{\tau_n,k}^{(1)}(f) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_n} \left\{ l_{n,k}(V_i) 1_{B_{n,k}(f)}(V_i) - E_{i-1}(l_{n,k}(V_i) 1_{B_{n,k}(f)}(V_i)) \right\}.$$

Then, since $\theta \le \frac{a_0}{2}$,

$$S_{\tau_n}^{(\theta)}(f) = \sum_{k=0}^{k_n+1} S_{\tau_n,k}(f).$$

For $0 \le k \le k_n$,

(7)
$$\begin{aligned}
&|S_{\tau_n,k}(f) - L_{\tau_n,k}^{(1)}(f)| \\
\le& \frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_n} \left\{ f^{(\sqrt{n}\theta)}(V_i) - l_{n,k}(V_i) \right\} 1_{B_{n,k}(f)}(V_i) \\
&+ \frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_n} E_{i-1} \left(\{ f^{(\sqrt{n}\theta)}(V_i) - l_{n,k}(V_i)\} 1_{B_{n,k}(f)}(V_i) \right) \\
\le& \frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_n} \{u_{n,k}(V_i) - l_{n,k}(V_i)\} 1_{A_{n,k}(f)}(V_i) \\
&+ \frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_n} E_{i-1}(\{u_{n,k}(V_i) - l_{n,k}(V_i)\} 1_{A_{n,k}(f)}(V_i)) \\
:=& R_{\tau_n,k}^{(1)}(f) + R_{\tau_n,k}^{(0)}(f).
\end{aligned}$$

Likewise, we have

(8)
$$\begin{aligned}
&\left| S_{\tau_n,k_n+1}(f) - L_{\tau_n,k_n+1}^{(1)}(f) \right| \\
\le& \frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_n} \left\{ u_{n,k_n+1}(V_i) - l_{n,k_n+1}(V_i) \right\} 1_{B_{n,k_n+1}(f)}(V_i) \\
&+ \frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_n} E_{i-1} \left(\{ u_{n,k_n+1}(V_i) - l_{n,k_n+1}(V_i)\} 1_{B_{n,k_n+1}(f)}(V_i) \right) \\
:=& R_{\tau_n,k_n+1}^{(1)}(f) + R_{\tau_n,k_n+1}^{(0)}(f).
\end{aligned}$$

Note that, on the set $B_{n,k_n+1}(f)$, we have

$$\frac{u_{n,k_n+1}(V_i) - l_{n,k_n+1}(V_i)}{\sqrt{n}} \le a_{k_n+1} \le \frac{(J(\delta)+\eta\delta)\sqrt{D}}{n}.$$

So we have

$$R^{(1)}_{\tau_n,k_n+1}(f) \le (J(\delta)+\eta\delta)\sqrt{D},$$

and

$$R^{(0)}_{\tau_n,k_n+1}(f) \le (J(\delta)+\eta\delta)\sqrt{D}.$$

Step 5 : Now, on the individual $B_{n,k}(f)'s$, we compare each lower δ_k- approximation, $l_{n,k}$, to the lower δ_0-approximation, $l_{n,0}$. For each $f \in \mathcal{F}$, let

$$L^{(0)}_{\tau_n}(f) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_n} \{l_{n,o}(V_i) - E_{i-1}(l_{n,0}(V_i))\}.$$

For $0 \le k \le k_n + 1$, let

$$L^{(0)}_{\tau_n,k}(f) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_n} \left\{l_{n,0}(V_i)1_{B_{n,k}(f)}(V_i) - E_{i-1}(l_{n,0}(V_i)1_{B_{n,k}(f)}(V_i))\right\}$$

so that $L^{(0)}_{\tau_n}(f) = \sum_{k=0}^{k_n+1} L^{(0)}_{\tau_n,k}(f)$. Note that $L^{(0)}_{\tau_n,0}(f) - L^{(1)}_{\tau_n,0}(f) = 0$, and for $1 \le k \le k_n+1$, $l_{n,k}(\cdot) - l_{n,0}(\cdot) = \sum_{j=1}^{k}(l_{n,j}(\cdot) - l_{n,j-1}(\cdot))$. Therefore we have

$$\begin{aligned}
&\sum_{k=0}^{k_n+1} \left(L^{(1)}_{\tau_n,k}(f) - L^{(0)}_{\tau_n,k}(f)\right) \\
&= \sum_{k=1}^{k_n+1} \frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_n} \sum_{j=1}^{k} \{((l_{n,j}(V_i) - l_{n,j-1}(V_i))1_{B_{n,k}(f)}(V_i)) \\
&\quad - E_{i-1}((l_{n,j}(V_i) - l_{n,j-1}(V_i))1_{B_{n,k}(f)}(V_i)))\} \\
&= \sum_{j=1}^{k_n+1} \frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_n} \{((l_{n,j}(V_i) - l_{n,j-1}(V_i))1_{C_{n,j}(f)}(V_i)) \\
&\quad - E_{i-1}((l_{n,j}(V_i) - l_{n,j-1}(V_i))1_{C_{n,j}(f)}(V_i)))\} \\
&:= \sum_{j=1}^{k_n+1} R^{(2)}_{\tau_n,j}(f).
\end{aligned} \tag{9}$$

Step 6 : We now compare $S_{\tau_n}^{(\theta)}$ to $L_{\tau_n}^{(0)}$ defined above. Combining (7),(8) and (9), we have for each $f \in \mathcal{F}$, and for $\theta \leq \frac{a_0}{2}$,

$$\left|S_{\tau_n}^{(\theta)}(f) - L_{\tau_n}^{(0)}(f)\right| = \left|\sum_{k=0}^{k_n+1} \{S_{\tau_n,k}(f) - L_{\tau_n,k}^{(0)}(f)\}\right|$$
$$\leq \left|\sum_{k=0}^{k_n+1} \{S_{\tau_n,k}(f) - L_{\tau_n,k}^{(1)}(f)\}\right| + \left|\sum_{k=0}^{k_n+1} \{L_{\tau_n,k}^{(1)}(f) - L_{\tau_n,k}^{(0)}(f)\}\right|$$
$$\leq \sum_{k=0}^{k_n+1} R_{\tau_n,k}^{(0)}(f) + \sum_{k=0}^{k_n+1} R_{\tau_n,k}^{(1)}(f) + \sum_{k=1}^{k_n+1} |R_{\tau_n,k}^{(2)}(f)|.$$

Therefore we have

$$||S_{\tau_n}^{(\theta)}||_\delta \leq ||L_{\tau_n}^{(0)}||_\delta + 2||S_{\tau_n}^{(\theta)} - L_{\tau_n}^{(0)}||_\delta$$
$$\leq ||L_{\tau_n}^{(0)}||_\delta + 2\sum_{k=0}^{k_n} ||R_{\tau_n,k}^{(0)}|| + 2\sum_{k=0}^{k_n} ||R_{\tau_n,k}^{(1)}|| + 2\sum_{k=1}^{k_n+1} ||R_{\tau_n,k}^{(2)}||.$$

When η_0, $\{\eta_k^{(0)} : 0 \leq k \leq k_n\}$,$\{\eta_k^{(1)} : 0 \leq k \leq k_n\}$ and $\{\eta_k^{(2)} : 1 \leq k \leq k_n + 1\}$ are constants which satisfy

$$2\eta_0 + 2\sum_{k=0}^{k_n} \eta_k^{(0)} + 2\sum_{k=0}^{k_n} \eta_k^{(1)} + 4\sum_{k=1}^{k_n+1} \eta_k^{(2)} \leq K(J(\delta) + \eta\delta)\sqrt{D} \tag{10}$$

for a positive constant K, we have

$$P\{||S_{\tau_n}^{(\theta)}||_\delta > (K+4)(J(\delta) + \eta\delta)\sqrt{D}\}$$
$$\leq P\left\{||L_{\tau_n}^{(0)}||_\delta > 2\eta_0\right\}$$
$$+ \sum_{k=0}^{k_n} P\left\{||R_{\tau_n,k}^{(0)}|| > \eta_k^{(0)}\right\}$$
$$+ \sum_{k=0}^{k_n} P\left\{||R_{\tau_n,k}^{(1)}|| > \eta_k^{(1)}\right\}$$
$$+ \sum_{k=1}^{k_n+1} P\left\{||R_{\tau_n,k}^{(2)}|| > 2\eta_k^{(2)}\right\}. \tag{11}$$

The values of the constants $\eta_0, \eta_k^{(0)}, \eta_k^{(1)}$, and $\eta_k^{(2)}$ will be specified later.

Step 7 : The individual terms of the equation (11) above are bounded using Freedman inequality and the upper bound of the cardinality of $\bigcup_{j=0}^{k} \mathcal{F}(\delta_j)$ where

$$\mathcal{F}(\delta) := \{f_{0,\delta}^l, f_{0,\delta}^u, \cdots, f_{\nu^B(\delta),\delta}^l, f_{\nu^B(\delta),\delta}^u\}.$$

Fix $f \in \mathcal{F}$. Take $\eta_k^{(0)} = D\frac{\delta_k^2}{a_{k+1}}$. Then we have

$$\begin{aligned}
&P\{R_{\tau_n,k}^{(0)}(f) \geq \eta_k^{(0)}\} \\
=&P\{a_{k+1}R_{\tau_n,k}^{(0)}(f) \geq D\delta_k^2\} \\
=&P\left\{a_{k+1}\frac{1}{\sqrt{n}}\sum_{i=1}^{\tau_n} E_{i-1}(\{u_{n,k}(V_i) - l_{n,k}(V_i)\}1_{A_{n,k}(f)}(V_i)) \geq D\delta_k^2\right\} \\
\leq&P\left\{\frac{1}{n}\sum_{i=1}^{\tau_n} E_{i-1}[u_{n,k}(V_i) - l_{n,k}(V_i)]^2 \geq D\delta_k^2\right\} \\
\leq&P\left\{\frac{1}{n}\sum_{i=1}^{\tau_n} E_{i-1}[f_{\delta_k}^u(V_i) - f_{\delta_k}^l(V_i)]^2 \geq D\delta_k^2\right\} \\
\leq&P\left\{\sum_{i=1}^{\tau_n} \frac{E_{i-1}[f_{\delta_k}^u(V_i) - f_{\delta_k}^l(V_i)]^2}{nd^2(f_{\delta_k}^l, f_{\delta_k}^u)} \geq D\right\} \\
\leq&P\left\{\sup_{f,g\in H_0}\sum_{i=1}^{\tau_n} \frac{E_{i-1}[f(V_i) - g(V_i)]^2}{nd^2(f,g)} \geq D\right\} \\
=&0
\end{aligned} \tag{12}$$

where we used (4) in the last equality.

Since $R_{\tau_n,k}^{(0)}$ depends on $\mathcal{F}$ only through the (at most) $\exp\{2\gamma_k\}$ members of $\bigcup_{j=0}^{k} \mathcal{F}(\delta_j)$, we have

$$\sum_{k=0}^{k_n} P\left\{||R_{\tau_n,k}^{(0)}|| > \eta_k^{(0)}\right\} \leq \sum_{k=0}^{k_n} \exp\{2\gamma_k\}||P\left\{R_{\tau_n,k}^{(0)}(\cdot) > \eta_k^{(0)}\right\}|| = 0.$$

Step 8 : The proof of the following Lemma 1 appears in Freedman (1975).

LEMMA 1. (Freedman Inequality) *Let* $(d_i)_{1\le i\le n}$ *be a martingale difference with respect to an increasing* σ*-fields* $(\mathcal{F}_i)_{0\le i\le n}$, *i.e.* $E(d_i|\mathcal{F}_{i-1}) = 0, i = 1,\cdots,n$. *Suppose that* $\|d_i\|_\infty \le M$ *for a constant* $M < \infty, i = 1,\cdots,n$. *Let* $\tau \le n$ *be a stopping time relative to the* $(\mathcal{F}_i)$ *that satisfies* $\|\sum_{i=1}^{\tau} E(d_i^2|\mathcal{F}_{i-1})\|_\infty \le L$ *for a constant* L. *Then for each* $\epsilon > 0$

$$P\left\{|\sum_{i=1}^{\tau} d_i| > \epsilon\right\} \le 2\exp\left\{-\frac{\epsilon^2}{2(L+M\epsilon)}\right\}.$$

For $0 \le k \le k_n$, $R^{(1)}_{\tau_n,k}(f)$ is a sum of nonnegative random variables each bounded by a_k, and note that $R^{(1)}_{\tau_n,k}(f) \le \eta_k^{(0)}$ a.s.. Note that

$$\begin{aligned}
&R^{(1)}_{\tau_n,k}(f) - R^{(0)}_{\tau_n,k}(f)\\
=&\frac{1}{\sqrt{n}}\sum_{i=1}^{\tau_n}\{\{u_{n,k}(V_i) - l_{n,k}(V_i)\}1_{A_{n,k}(f)}(V_i)\\
&- E_{i-1}(\{u_{n,k}(V_i) - l_{n,k}(V_i)\}1_{A_{n,k}(f)}(V_i))\}\\
:=&\sum_{i=1}^{\tau_n}\alpha_i, \text{ say,}
\end{aligned}$$

where $E_{i-1}(\alpha_i) = 0$. Note also that $|\alpha_i| \le 2a_k$ a.s.. Using the algebraic inequality $(a-b)^2 \le 2(a^2+b^2)$ and noting the calculation of (12) we see that

$$\sum_{i=1}^{\tau_n} E_{i-1}(\alpha_i^2) \le \frac{2}{n}\sum_{i=1}^{\tau_n} E_{i-1}[(u_{n,k}(V_i) - l_{n,k}(V_i)|^2 \le 2D\delta_k^2.$$

Take $\eta_k^{(1)} = 2\eta_k^{(0)}$. Note that $D\delta_k^2 = a_{k+1}\eta_k^{(0)} \le a_k\eta_k^{(0)}$. Then by Lemma 1 with $L = 2D\delta_k^2$, for each $f \in \mathcal{F}$,

$$\begin{aligned}
&P\left\{R^{(1)}_{\tau_n,k}(f) > \eta_k^{(1)}\right\}\\
\le&P\left\{R^{(1)}_{\tau_n,k}(f) - R^{(0)}_{\tau_n,k}(f) > \eta_k^{(1)} - \eta_k^{(0)}\right\}\\
=&P\left\{\sum_{i=1}^{\tau_n}\alpha_i > \eta_k^{(0)}\right\} \le \exp\left\{-\frac{\eta_k^{(0)2}}{2(2D\delta_k^2 + 2a_k\eta_k^{(0)})}\right\}\\
\le&\exp\left\{-\frac{\eta_k^{(0)2}}{2(2a_k\eta_k^{(0)} + 2a_k\eta_k^{(0)})}\right\} \le \exp\left\{-\frac{\eta_k^{(0)}}{8a_k}\right\}.
\end{aligned}$$

Hence, since $R^{(1)}_{\tau_n,k}$ depends on $\mathcal{F}$ only through the (at most) $\exp\{2\gamma_k\}$ members of $\bigcup_{j=0}^{k}\mathcal{F}(\delta_j)$,

$$\sum_{k=0}^{k_n} P\{\|R^{(1)}_{\tau_n,k}\| > \eta_k^{(1)}\} \le \sum_{k=0}^{k_n} \exp\{2\gamma_k\}\|P\{R^{(1)}_{\tau_n,k}(\cdot) > \eta_k^{(1)}\}\|$$
$$\le \sum_{k=0}^{k_n} \exp\left\{2\gamma_k - \frac{\eta_k^{(0)}}{8a_k}\right\} \le \sum_{k=0}^{k_n} \exp\left\{2\gamma_k - \frac{D\delta_k^2}{8a_k a_{k+1}}\right\}$$
$$\le \sum_{k=0}^{k_n} \exp\left\{2\gamma_k - \frac{D\delta_k^2}{8a_k^2}\right\} \le \sum_{k=0}^{\infty} \exp\{-\eta^2 Lk\}$$

where

$$a_k = \delta_k \left(\frac{D}{8(2\gamma_k + \eta^2 Lk)}\right)^{1/2}. \tag{13}$$

Note that the strictly decreasing sequence $\{a_k\}$ in (13) is chosen so that

$$2\gamma_k - \frac{D\delta_k^2}{8a_k^2} = -\eta^2 Lk.$$

Step 9 : Note that for $1 \le k \le k_n + 1$, $f \in \mathcal{F}$, $R^{(2)}_{\tau_n,k}(f)$ is a sum of martingale difference sequence. So we may write

$$R^{(2)}_{\tau_n,k}(f) := \sum_{i=1}^{\tau_n} \beta_i, \ say,$$

where $E_{i-1}(\beta_i) = 0$. By (6), we have $|\beta_i| \le 2a_k$ a.s., and by the similar argument as in *Step* 8, we get

$$\sum_{i=1}^{\tau_n} E_{i-1}(\beta_i^2) \le \frac{2}{n}\sum_{i=1}^{\tau_n} E_{i-1}[(u_{n,k-1}(V_i) - l_{n,k-1}(V_i)]^2 \le 2D\delta_{k-1}^2.$$

Take $\eta_k^{(2)} = \frac{D\delta_{k-1}^2}{a_k}$. Note that $\eta_k^{(2)} = \eta_{k-1}^{(0)}$, and $\delta_k^2 < \delta_{k-1}^2$.

Again by Lemma 1 with $L = 2D\delta_{k-1}^2$, for each $f \in \mathcal{F}$, we have

$$P\left\{R_{\tau_n,k}^{(2)}(f) > \eta_k^{(2)}\right\} = P\left\{\sum_{i=1}^{\sigma_n(1)} \beta_i > \eta_k^{(2)}\right\}$$
$$\leq 2\exp\left\{-\frac{\eta_k^{(2)^2}}{2(2D\delta_{k-1}^2 + 2a_k\eta_k^{(2)})}\right\} = 2\exp\left\{-\frac{\eta_k^{(2)^2}}{2(2a_k\eta_k^{(2)} + 2a_k\eta_k^{(2)})}\right\}$$
$$= 2\exp\left\{-\frac{D\delta_{k-1}^2}{8a_k^2}\right\} \leq 2\exp\left\{-\frac{D\delta_k^2}{8a_k^2}\right\}.$$

Note that $R_{\tau_n,k}^{(2)}$ also depends on $\mathcal{F}$ only through the (at most) $\exp\{2\gamma_k\}$ members of $\bigcup_{j=0}^{k} \mathcal{F}(\delta_j)$. So we have

$$\sum_{k=1}^{k_n+1} P\left\{||R_{\tau_n,k}^{(2)}|| > \eta_k^{(2)}\right\}$$
$$\leq \sum_{k=1}^{k_n+1} \exp\left\{2\gamma_k - \frac{D\delta_k^2}{8a_k^2}\right\} \leq 2\sum_{k=0}^{\infty} \exp\{-\eta^2 Lk\}$$

Step 10 : Finally, for $f, g \in \mathcal{F}$, note that

$$L_{\tau_n}^{(0)}(f) - L_{\tau_n}^{(0)}(g)$$
$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_n} \left\{l_{n,o}^f(V_i) - l_{n,o}^g(V_i) - E_{i-1}(l_{n,0}^f(V_i) - l_{n,o}^g(V_i))\right\}$$

where $l_{n,o}^f$ and $l_{n,o}^g$ are $l_{n,o}$ corresponding to f and g respectively. We write, as before,

$$L_{\tau_n}^{(0)}(f) - L_{\tau_n}^{(0)}(g) := \sum_{i=1}^{\tau_n} \zeta_i, \ say,$$

where $E_{i-1}(\zeta_i) = 0$. Note that $|\zeta_i| \leq 2a_0$ a.s.. When $d(f,g) < \delta$, we have

$$\frac{1}{n}\sum_{i=1}^{\tau_n} E_{i-1}(l_{n,0}^f(V_i) - l_{n,o}^g(V_i))^2 \leq \frac{1}{n}\sum_{i=1}^{\tau_n}[\{E_{i-1}(f(V_i) - g(V_i))^2\}^{1/2}$$
$$+ \{E_{i-1}(u_{n,o}^f(V_i) - l_{n,o}^f(V_i))^2\}^{1/2} + \{E_{i-1}(u_{n,o}^g(V_i) - l_{n,o}^g(V_i))^2\}^{1/2}]^2$$

$$\leq \frac{1}{n}\sum_{i=1}^{\tau_n}[\{E_{i-1}(f(V_i)-g(V_i))^2\}^{1/2} + \{E_{i-1}(f^u_{\delta_0}(V_i)-f^l_{\delta_0}(V_i))^2\}^{1/2}$$
$$+ \{E_{i-1}(g^u_{\delta_0}(V_i)-g^l_{\delta_0}(V_i))^2\}^{1/2}]^2$$
$$\leq \frac{3}{n}\sum_{i=1}^{\tau_n} E_{i-1}[f(V_i)-g(V_i)]^2 + \frac{3}{n}\sum_{i=1}^{\tau_n} E_{i-1}[f^u_{\delta_0}(V_i)-f^l_{\delta_0}(V_i)]^2$$
$$+ \frac{3}{n}\sum_{i=1}^{\tau_n} E_{i-1}[g^u_{\delta_0}(V_i)-g^l_{\delta_0}(V_i)]^2 \leq 3(Dd^2(f,g)+D\delta^2+D\delta^2)\ a.s.$$
$$\leq 9D\delta^2\ a.s..$$

In the third inequality above we used the algebraic inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$. So we have

$$\sum_{i=1}^{\tau_n} E_{i-1}(\zeta_i^2) \leq \frac{2}{n}\sum_{i=1}^{\tau_n} E_{i-1}[(l^f_{n,0}(V_i) - l^g_{n,0}(V_i)]^2 \leq 18D\delta^2.$$

Take $\eta_0 = \frac{9D\delta^2}{a_0}$. By Lemma 1 with $L = 18D\delta^2$,

$$P\left\{L^{(0)}_{\tau_n}(f) - L^{(0)}_{\tau_n}(g) > \eta_0\right\}$$
$$=P\left\{\sum_{i=1}^{\tau_n}\zeta_i > \eta_0\right\}$$
$$\leq 2\exp\left\{-\frac{{\eta_0}^2}{2(18D\delta^2+2a_0\eta_0)}\right\}.$$

Therefore we have

$$P\{||L^{(0)}_{\tau_n}||_\delta > \eta_0\}$$
$$\leq 2\exp\left\{4\gamma_0 - \frac{{\eta_0}^2}{2(18D\delta^2+2a_0\eta_0)}\right\}$$
$$\leq 2\exp\left\{4\gamma_0 - \frac{{\eta_0}^2}{2(2a_0\eta_0+2a_0\eta_0)}\right\}$$
$$=2\exp\left\{-14\gamma_0 - 9\eta^2\right\}$$
$$\leq 2\exp\{-\eta^2\}$$
$$\leq 2\sum_{k=0}^{\infty}\exp\{-\eta^2 Lk\}.$$

By (11), it remains to show that (10) holds for some fixed constant K. Note that

$$\begin{aligned}
\sum_{k=0}^{k_n} \eta_k^{(0)} =& D \sum_{k=0}^{k_n} \frac{\delta_k^2}{a_{k+1}} \\
\leq& D \sum_{k=0}^{\infty} \frac{\delta_k^2 8(2\gamma_{k+1} + \eta^2 L(k+1))^{1/2}}{\delta_{k+1} D^{1/2}} \\
=& 32 D^{1/2} \sum_{k=1}^{\infty} \delta_k (2\gamma_k + \eta^2 Lk)^{1/2} \\
\leq& 32 D^{1/2} \sum_{k=1}^{\infty} \delta_k (2\gamma_k^{1/2} + \eta L^{1/2} k) \\
=& 32(2D)^{1/2} \sum_{k=1}^{\infty} \delta_k \gamma_k^{1/2} + 32 D^{1/2} \eta \sum_{k=1}^{\infty} \delta_k L^{1/2} k.
\end{aligned}$$

We write $\sum_{k=0}^{\infty} \delta_k L^{1/2} k := \tilde{c}\delta$, where $\tilde{c} = \sum_{k=0}^{\infty} \frac{L^{1/2} k}{2^k}$. By definition of γ_k,

$$\begin{aligned}
&\sum_{k=1}^{\infty} \delta_k \gamma_k^{1/2} \leq \sum_{k=1}^{\infty} \delta_k \sum_{j=k}^{\infty} [H^B(\delta_j)]^{1/2} \\
\leq& \sum_{j=0}^{\infty} [H^B(\delta_j)]^{1/2} \sum_{k=j}^{\infty} \delta_k = 2 \sum_{j=0}^{\infty} \delta_j [H^B(\delta_j)]^{1/2} \\
\leq& 4 \sum_{j=0}^{\infty} \int_{\delta_{j+1}}^{\delta_j} [H^B(u)]^{1/2} du = 4J(\delta).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\sum_{k=0}^{k_n} \eta_k^{(0)} \leq& 4 \cdot 32(2D)^{1/2} J(\delta) + 32 D^{1/2} \eta \tilde{c} \delta \\
=& (128\sqrt{2} + 32\tilde{c})(J(\delta) + \eta\delta)\sqrt{D}.
\end{aligned}$$

Recall that $\eta_k^{(1)} = \eta_k^{(0)}$, and $\eta_k^{(2)} = \eta_{k-1}^{(0)}$. Then we have

$$\begin{aligned}\eta_0 =& 9\sqrt{8}\sqrt{D}\delta(2\gamma_0 + \eta^2)^{1/2}\\ \leq& 36\sqrt{D}(\delta\gamma_0^{1/2} + \eta\delta)\\ =& 36\sqrt{D}(\delta[H^B(u)]^{1/2} + \eta\delta)\\ \leq& 36(J(\delta) + \eta\delta)\sqrt{D}.\end{aligned}$$

Therefore we have

$$\begin{aligned}& 2\eta_0 + 2\sum_{k=0}^{k_n}\eta_k^{(0)} + 2\sum_{k=0}^{k_n}\eta_k^{(1)} + 4\sum_{k=1}^{k_n+1}\eta_k^{(2)}\\ =& 2\eta_0 + 2\sum_{k=0}^{k_n}\eta_k^{(0)} + 4\sum_{k=0}^{k_n}\eta_k^{(0)} + 4\sum_{k=1}^{k_n+1}\eta_k^{(0)}\\ \leq& K(J(\delta) + \eta\delta)\sqrt{D}\end{aligned}$$

where $K = 72 + 128\sqrt{2} + 320\tilde{c}$. We have shown that (10) holds. This completes the proof of Proposition 1.

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