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ON THE STRUCTURE OF DISCRETE SPECTRUM OF THE NON–SELFADJOINT SYSTEM OF DIFFERENTIAL EQUATIONS IN THE FIRST ORDER

ÖMER AKIN AND ELGIZ BAIRAMOV

1. Introduction

This paper is concerned with the problem given below

$$(1.1)\qquad \begin{aligned} &i\frac{du_1(x,\lambda)}{dx} + q_1(x)u_2(x,\lambda) = \lambda u_1(x,\lambda) \qquad 0 \le x < \infty \\ &-i\frac{du_2(x,\lambda)}{dx} + q_2(x)u_1(x,\lambda) = \lambda u_2(x,\lambda), \end{aligned}$$

$$(1.2)\qquad u_2(0,\lambda) - hu_1(0,\lambda) = 0$$

where λ is a complex parameter and h is a non-zero complex number.

By applying the transformation of

$$y_1(x,\lambda) = \frac{1}{2}[u_1(x,\lambda) + u_2(x,\lambda)], \quad y_2(x,\lambda) = \frac{1}{2i}[u_2(x,\lambda) - u_1(x,\lambda)]$$

to the system (1.1) we can see that it has become the following

$$(1.3)\qquad B\frac{dy(x,\lambda)}{dx} + \Omega(x)y(x,\lambda) = \lambda y(x,\lambda)$$

Where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad y(x,\lambda) = \begin{pmatrix} y_1(x,\lambda) \\ y_2(x,\lambda) \end{pmatrix}, \quad \Omega(x) = \begin{pmatrix} \alpha(x), & \beta(x) \\ \beta(x), & -\alpha(x) \end{pmatrix}$$

$$\alpha(x) = \frac{1}{2}[q_1(x) + q_2(x)], \qquad \beta(x) = \frac{i}{2}[q_1(x) - q_2(x)]$$

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This equation (1.3) is called the canonical Dirac system of which mass is zero.

We have already known that some problems which are very close to this problem has been treated by some authors, such as in the works [8, 9], the structure of the discrete spectrum of non-self adjoint Schrödinger operator is examined. The eigen values and spectral singularities of the dissipative Schrödinger operators with rapidly decreasing potential were investigated in [4]. In the work [7] has been proved that the eigenvalues and the spectral singularities of the problem (1.1)-(1.2) are finite under the conditions

$$(1.4) \qquad |q_i(x)| \leq \ Ce^{\epsilon x}, \quad \epsilon > 0, \quad i = 1, 2, C \text{ constant.}$$

In the present work, we have proved that the eigenvalues and the spectral singularities of the problem, given by (1.1)-(1.2), are finite under the following conditions

$$(1.5) \qquad |q_i(x)| \leq \ Ce^{\epsilon\sqrt{x}}, \quad \epsilon > 0, \quad i = 1, 2$$

Our proof is different from the proof used in [7]. The reason for that is the proof of [7] is not applicable in the case of (1.5). It can easily be seen that the conditions (1.5) are weaker than (1.4).

Furthermore when the functions q_i, $i = 1, 2$, are decreasing as polynomials the structure of spectral sets of the operator L are also examined in this work.

2. On the solutions of the system (1.1)

Let us define the operator L on the space of

$$L_2(0, \infty, C_2) := \left\{ f : f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad \int_0^\infty (|f_1|^2 + |f_2|^2) dx < \infty \right\}$$

with the help of (1.1) and (1.2) as follows: For $u \in L_2(0, \infty; C_2)$, which are differentiable,

$$1(u) := \begin{pmatrix} iu_1' + 9_1 u_2 \\ -iu_2' + 9_2 u_1 \end{pmatrix}.$$

$D(L) := \{u : u \in L_2(0,\infty;C_2),\ u$ absolutely continuous, $l(u) \in L_2(0,\infty;C_2),\ u_2(0) - hu_1(0) = 0\}$

Then if $u \in D(L)$ we define $Lu := l(u)$

Suppose that $q_1, q_2 \in C(0,\infty)$ and satisfy the inequality

$$|q_1(x)| \le C,\ (1+x)^{-(1+\epsilon)},\quad i = 1,2, \tag{2.1}$$

where $C > 0, \epsilon > 0$.

Let $q_i(x) \equiv 0,\ i = 1,2$, the equation (1.1) has the solution given below

$$f(x,\lambda) = \begin{pmatrix} A \cdot \exp(-i\lambda x) \\ B \cdot \exp(i\lambda x) \end{pmatrix}$$

A, B are constants.

Let λ be a real number and $u(x,\lambda)$ is a bounded solution of (1.1) with the following conditions

$$u_1(x,\lambda) = \mathrm{A}e^{-i\lambda x} + o(1),\quad u_2(x,\lambda) = \mathrm{B}e^{i\lambda x} + o(1),\quad x \to \infty \tag{2.2}$$

In [2], [7] have been shown that the solution (2.2) has the following form:

(2.3)

$$u_1(x,\lambda) = \mathrm{A}e^{-i\lambda x} + A\int_x^\infty H_{11}(x,s)e^{-i\lambda s}ds + B\int_x^\infty H_{12}(x,s)e^{i\lambda s}ds$$

$$u_2(x,\lambda) = \mathrm{B}e^{i\lambda x} + A\int_x^\infty H_{21}(x,s)e^{-i\lambda s}ds + B\int_x^\infty H_{22}(x,s)e^{i\lambda s}ds$$

where $H_{i,j}(x,s),\ i,j = 1,2$, are the solutions of the following two systems of Volterra integral equations;

$$\begin{aligned} H_{11}(x,s) &= -i\int_x^\infty H_{21}(t,t+s-x)q_1(t)dt,\quad 0 \le x \le s \\ H_{21}(x,s) &= \frac{-1}{2i}q_2\left(\frac{x+s}{2}\right) + i\int_x^{\frac{x+s}{2}} H_{11}(t,x+s-t)q_2(t)dt, \end{aligned} \tag{2.4}$$

$$\begin{aligned} H_{22}(x,s) &= i\int_x^\infty H_{12}(t,t+s-x)q_2(t)dt,\quad 0 \le x \le s \\ H_{12}(x,s) &= \frac{1}{2i}q_1\left(\frac{x+s}{2}\right) + \frac{1}{i}\int_x^{\frac{x+s}{2}} H_{22}(t,x+s-t)q_1(t)dt \end{aligned} \tag{2.5}$$

In the works [2] and [7] have been shown that these systems (2.4) and (2.5), have the bounded solutions which are unique. If the functions $q_i(x)$, $i = 1, 2$, satisfy the estimate (2.1) then for the functions $H_{ij}(x, s)$, $0 \le x \le s$, $i, j = 1, 2$, the estimate given below holds good

$$|H_{i,j}(x,s)| \le C, \quad (1+x+s)^{-(1+\epsilon)} \tag{2.6}$$

The matrix function

$$H(x,s) := \begin{pmatrix} H_{11}(x,s) & H_{12}(x,s) \\ H_{21}(x,s) & H_{22}(x,s) \end{pmatrix}$$

has the role of the kernel of the operator transformation in the quantum scattering theory [5].

Let consider the vector functions

$$e^{(1)}(x,\lambda) = \begin{pmatrix} e_1^{(1)}(x,\lambda) \\ e_2^{(1)}(x,\lambda) \end{pmatrix} = \begin{pmatrix} e^{-i\lambda x} + e^{-i\lambda x} \int_0^\infty H_{11}(x, x+t) e^{-i\lambda t} dt \\ e^{-i\lambda x} \int_0^\infty H_{21}(x, x+t) e^{-i\lambda t} dt \end{pmatrix}$$

$$e^{(2)}(x,\lambda) = \begin{pmatrix} e_1^{(2)}(x,\lambda) \\ e_2^{(2)}(x,\lambda) \end{pmatrix} = \begin{pmatrix} e^{i\lambda x} \int_0^\infty H_{12}(x, x+t) e^{i\lambda t} dt \\ e^{i\lambda x} + e^{i\lambda x} \int_0^\infty H_{22}(x, x+t) e^{i\lambda t} dt \end{pmatrix}$$

Where λ is a real number and $H_{i,j}(x, x+t)$, $x, t \ge 0$, $i, j = 1, 2$, were given as the solutions of (2.4) and (2.5). We can easily see that the vector functions $e^{(1)}$ and the $e^{(2)}$ are the proper cases of the system of (2.3) i.e. these are the solutions of (1.1). Since the Wronskian $W[e^{(1)}, e^{(2)}]$ is independent of x we can take the limit of W as $x \to +\infty$, we see that $\lim_{x\to\infty} W[e^{(1)}(x,\lambda),\ e^{(2)}(x,\lambda)] = 1$ Therefore these solutions are the fundamental solutions of the system (1.1).

It is clear from the definition of $e^{(1)}(x,\lambda)$ and the relation (2.6) that the function $e^{(1)}(x,\lambda)$ has an analytic continuation on λ to the lower half-plane. The similar result is also true for the function $e^{(2)}(x,\lambda)$ to the upper half-plane.

$$\begin{aligned} D_+(\lambda) &:= e_2^{(2)}(0,\lambda) - \mathrm{h}e_1^{(2)}(0,\lambda) \\ D_-(\lambda) &:= e_2^{(1)}(0,\lambda) - \mathrm{h}e_1^{(1)}(0,\lambda) \end{aligned} \tag{2.7}$$

Definition 2.1. The roots of $D_+(\lambda) = 0$ on the upper plane, $\mathrm{Im}\lambda > 0$, and the roots of $D_-(\lambda) = 0$ on the lower plane, $\mathrm{Im}\lambda < 0$ are called the singular numbers of the operator L. Multiplicity of the root is called the multiplicity of that singular number of the operator L.

Lemma 2.1. *The singular numbers of the operator L are identical when its non-real eigenvalues and the multiplicity of that singular numbers are also identical with the corresponding eigenvalues.*

Proof. We only prove the first part of the lemma, because the second part is easily obtained from the definition 2.1. Suppose that λ_k, $\mathrm{Im}\lambda_k > 0$, be a singular number of L, then $D_+(\lambda_k) = 0$, the function $e^{(2)}(x, \lambda_k)$ is a member of the space $L_2(0, \infty; C_2)$ and it is a solution of the problem given by the formulae (1.1) and (1.2). Therefore $e^{(2)}(x, \lambda_k)$ is the eigen function of the operator L corresponding to the eigenvalue λ_k. On the opposite side if λ_o, $\mathrm{Im}\lambda_o > 0$, is an arbitrary eigenvalue of L, then the problem (1.1)-(1.2) has a solution with the properties $u(x, \lambda_o) \not\equiv 0$, $u \in L_2(0, \infty; C_2)$. Under these conditions $u_1(0, \lambda_o)$ and $u_2(0, \lambda_o)$ both are non-zero numbers. Otherwise the uniqueness of the solution of the Cauchy problem related to (1.1) and (1.2) gives that $u(x, \lambda_o) \equiv 0$.

The Wronskian of $u(x, \lambda_o)$ and $e^{(2)}(x, \lambda_o)$ is not depending on x then

$$W[u, e^{(2)}] = \lim_{x\to\infty} \{u_1(x, \lambda_o)e_2^{(2)}(x, \lambda_o) - u_2(x, \lambda_o)e_1^{(2)}(x, \lambda_o)\} = 0$$

holds good and because of

$$\begin{aligned} W[u, e^{(2)}] &= u_1(0, \lambda_o)e_2^{(2)}(0, \lambda_o) - u_2(0, \lambda_o)e_1^{(2)}(0, \lambda_o) \\ &= u_1(0, \lambda_o)[e_2^{(2)}(0, \lambda_o) - \mathrm{h}e_1^{(2)}(0, \lambda_o)] = 0 \\ e_2^{(2)}(0,\lambda_o) &- \mathrm{h}e_1^{(2)}(0, \lambda_o) = 0 \end{aligned}$$

must be satisfied. That is to say $D_+(\lambda_o) = 0$ is obtained. This shows us that λ_o is a singular number of the operator L. We can show the result for $D_-(\lambda)$ with the similar way. This completes the proof.

By using [6] we can easily see that the zeros fo $D_+(\lambda)$ and $D_-(\lambda)$ on the real line are spectral singularities of the operator L. This result and lemma 2.1 show us that the structure of discrete spectrum of the operator L is the same as the structure of the roots of $D_+(\lambda) = 0$ on the closed upper plane and the roots of $D_-(\lambda) = 0$ on the closed lower plane. For the sake of shortness and having the same structure of the roots of $D_+(\lambda) = 0$ and $D_-(\lambda) = 0$ examine the structure of $D_+(\lambda)$ on the upper semi-plane.

3. The structure of the discrete spectrum of L

DEFINITION 3.1. If the complex valued functions $q_i(x)$, $i = 1,2$, continuous on the interval $(0,\infty)$ and

$$|q_i(x)| \le C_k.\ (1+x)^{-(k+1+\epsilon)} \tag{3.1}$$

is satisfied for $\epsilon > 0$, $k = 0,1,2,\cdots,n$, $i = 1,2$, $n \in Z^+$ for some constant C_k then the functions $q_i(x)$ are called in the class of $M_{n,\epsilon}$. Specially if the inequality (3.1) is satisfied for all n in $Z^+ U\{+\infty\}$ then we call $q_i(x)$ in the class of M_∞ where $C_k \in R^+$ constants and independent of x.

We give the following notations for the use of sequel.

$$\begin{aligned}
S^+ &:= \{\lambda :\ \mathrm{Im}\lambda \ge 0,\ \ D_+(\lambda) = 0\}\\
S_0^+ &:= \{\lambda :\ \mathrm{Im}\lambda > 0,\ \ D_+(\lambda) = 0\}\\
S_1^+ &:= \{\lambda :\ \mathrm{Im}\lambda = 0,\ \ D_+(\lambda) = 0\}\\
S_2^+ &:= \{\lambda : D_+(\lambda) = 0,\ \text{the multiplicity of } \lambda \text{ is infinite}\}\\
S_3^+ &:= \{\lambda : \exists(\lambda_n),\ \mathrm{Im}\lambda_n > 0,\ D_+(\lambda_n) = 0,\ \lambda_n \to \lambda,\ n \to \infty\}\\
&= \{\text{The limit points of eigenvalues of the operator } L\\
&\qquad \text{on the upper half-plane}\}
\end{aligned}$$

LEMMA 3.1. *If $q_i(x)$ in class $M_{k,\epsilon}$ then the integral*

$$B_k := \int_0^\infty t^k |H_{ij}(0,t)| dt,\quad i,j = 1,2 \tag{3.2}$$

is convergent and the function

$$D_+(\lambda) = e_2^{(2)}(0,\lambda) - h e_1^{(2)}(0,\lambda)$$

is analytic for Im$\lambda > 0$. Furtheremore all the derivatives up to k of $D_+(\lambda)$ are continuous for Im$\lambda \geq 0$. The derivatives, mentioned above, has the property given below

$$|D_+^{(k)}(\lambda)| \leq CB_k, \quad k \geq 1, \quad \text{Im}\lambda \geq 0 \tag{3.3}$$

Finally

$$\sup_{\text{Im}\lambda \geq 0} |D_+(\lambda)| < \infty \tag{3.4}$$

Proof. We know that if $q_i(x) \in M_{k,\epsilon}$, $i = 1, 2$, then with the help of [2] and the systems of (2.4)-(2.5)

$$|H_{ij}(x,s)| \leq C_k(1 + x + s)^{-(k+1+\epsilon)} \quad i,j = 1,2, \tag{3.5}$$

holds good. We can easily see that with the help of (3.5) the integrals (3.2) are convergent. Since the functions $e_1^{(2)}(0,\lambda)$ and $e_2^{(2)}(0,\lambda)$ are analytic for Im$\lambda > 0$, $D_+(\lambda)$ is analytic on the open upper-plane. Since the equality

$$\begin{aligned} D_+(\lambda) &= e_2^{(2)}(0,\lambda) - \text{h}e_1^{(2)}(0,\lambda) \\ &= 1 + \int_0^\infty H_{22}(0,t)e^{i\lambda t}dt - h\int_0^\infty H_{12}(0,t)e^{i\lambda t}dt \end{aligned}$$

holds (3.3) and (3.4) can be obtained easily. This completes the proof.

Now we are going to prove some theorems for the structure of the discrete spectrum for the operator L. It is clear that $S_0^+ \cap S_1^+ = \emptyset$, $S_0^+ \cup S_1^+ = S^+$ and for all points in S_0^+ are the eigenvalues with the finite multiplicity of the operator L.

THEOREM 3.1. *If $q_i(x) \in M_{0,\epsilon}$ then the set S_1^+ is closed, its Lebesgue measure is zero, $S_3^+ \subset S_1^+$ and $S_2^+ \subset S_1^+$ are satisfied.*

Proof. Let us consider

$$D_+(\lambda) = 1 + \int_0^\infty [H_{22}(0,t) - hH_{12}(0,t)]e^{i\lambda t}dt.$$

We can obtain the following result by using this relation;

$$(3.6) \qquad D_+(\lambda) = 1 + 0(1) \quad \text{as} \quad |\lambda| \to +\infty, \quad \text{Im}\lambda \geq 0$$

The asymtotic equality (3.6) gives us that the zeros of $D_+(\lambda)$ on the closed upper half-plane are in a bounded region. We know that $D_+(\lambda)$ is continuous up to the real axis, therefore S_1^+ is closed, $S_3^+ \subset S_1^+, S_2^+ \subset S_1^+$ can be obtained easily. By the uniqueness theorems of analytic functions the Lebesgue measure of S_1^+ is zero. (See [10]). This completes the proof.

THEOREM 3.2. *If $q_i(x) \in M_{1,\epsilon}$ then*

1) *The set of eigenvalues of the operator L satisfies*

$$\sum_{\vee} Im\lambda_{\vee} < \infty$$

Here we also regard the multiple eigenvalues as the sum of multiplicity numbers.

2) *The following condition holds.*

$$(3.7) \qquad \sum_{\vee} |1_{\vee}| l_n |1_{\vee}| > -\infty$$

where $|1_{\vee}|$ is the length of the complementary interval $l_{\vee}$ of the set S_1^+ and the sum of (3.7) has been taken on the bounded complementary intervals.

Proof. By the asymptotic equality (3.6) the non-real zeros $\{\lambda_{\vee}\}$ of the function $D_+(\lambda)$ on the open upper half-plane are in a bounded region and the limits of $\{\lambda_{\vee}\}$ lie on the real line, i.e.

$$\text{Im}\lambda_{\vee} \to 0 \quad \text{as} \quad \vee \to \infty,$$

By the Lemma 3.1. the function $D_+(\lambda)$ is a bounded analytic function in $\text{Im}\lambda > 0$, then the following factorization holds ([11]),

$$D_+(\lambda) = cB(\lambda), \qquad \text{Im}\lambda > 0$$

where c is constant, $|c| = 1$, $B(\lambda)$ is a Blaschke product and defined by

$$B(\lambda) = \Pi_{\vee} \frac{\lambda - \lambda_{\vee}}{\lambda - \bar{\lambda}_{\vee}} = \Pi_{\vee}\Big(1 - \frac{2\mathrm{Im}\lambda_{\vee}}{\lambda - \bar{\lambda}_{\vee}}\Big), \qquad \mathrm{Im}\lambda > 0.$$

In the Blaschke product $\lambda_{\vee}$ are all the zeros of $D_{+}(\lambda)$ on the open upper half-plane and in this product the multiple zeros are repeated of the multiplicity numbers.

Now, we can take the number $R > 0$ as big as all the zeros of $D_{+}(\lambda)$ on the open upper-half plane are stayed on the following semi-disc P;

$$P := \{\lambda : \ \mathrm{Im}\lambda > 0, \ \ |\lambda| \leq R\}$$

Since for all arbitrary λ belonging to P $\lim_{\vee\to\infty}|\lambda - \bar{\lambda}_{\vee}| \geq \mathrm{Im}\,\lambda \neq 0$ we have

$$\lim_{\vee\to\infty} \frac{\mathrm{Im}\lambda_{\vee}}{|\lambda - \bar{\lambda}_{\vee}|} = 0.$$

We also know that the for all λ in P Blasehke product is absolutely couvergent, therefore we have the following result;

$$\sum_{\vee} \frac{2\mathrm{Im}\,\lambda_{\vee}}{|\lambda - \bar{\lambda}_{\vee}|} < \infty$$

By the last inequality and by the statement $0 \neq |\lambda - \bar{\lambda}_{\vee}| \leq 2R$, which holds for all λ in P, we obtain that

$$\sum_{\vee} \mathrm{Im}\lambda_{\vee} < \infty$$

is satisfied.

By the hypothesis of the theorem $D'_{+}(\lambda)$ exists and continuous on the real line.

We can prove (3.7) by using the uniqueness theorem of Beurling [1]. We state the theorem for the readers convenience;

Suppose that the function $g(z)$ is analytic in the unit disc, continuous upto its boundary and it satisfies the Hölder's condition on the boundary. If the function g is zero on the set $X \subset [0, 2\pi]$ of which Lebesgue measure zero, and the condition

$$\sum_{\vee} |1_{\vee}| 1_{n} |1_{\vee}| = -\infty$$

holds then $g(z) \equiv 0$, Hence the proof is completed.

THEOREM 3.3. *Let $q_i(x)$ be in M_∞. Then the S_2^+ is closed, its Lebesgue measure zero and $S_3^+ \subset S_2^+$ satisfied. Furtheremore*

$$\int_0^\infty ln\ T(s)d\theta(S_2^+, s) > -\infty \tag{3.8}$$

is satisfied, where $T(s) = \inf_k \frac{B_k S^k}{k!}$, $\theta(S_2^+, s)$ the linear measure of the neighbourhood of S_2^+. [We know that B_k is given by (3.2)].

Proof. Since $q_i(x) \in M_\infty$ then $D_+(\lambda)$ is infinitely differentiable on the real axis. Moreover the function $D_+(\lambda)$ and its all derivatives are continuous upto the real axis. This implies that S_2^+ is closed, its Lebesgue measure zero and $S_3^+ \subset S_2^+$. To prove (3.8) we use the following uniqueness theorem of Pavlov [9]: Let $g(z)$ is analytic in the unit disc, continuous with all derivatives upto its boundary and

$$\text{Sup}_{|z|<1}|g^{(k)}(z)| \leq A_k, \quad k = 0, 1, 2, \cdots$$

If $g(e^{i\theta}) = 0$ on the set $F \subset [0, 2\pi]$ of which Lebesgue measure zero and the condition $\int_0^\infty \ln T(s)d\theta(F, S) = -\infty$, holds then $g(z) \equiv 0$. Hence the proof is completed.

THEOREM 3.4. *If the conditions*

$$\begin{aligned} &|q_i(x)| \leq C\ \exp(-\delta x^\alpha), \quad i = 1, 2, \\ &\qquad \delta > 0,\ 0 < \alpha < 1/2 \end{aligned} \tag{3.9}$$

is satisfied then S_2^+ is closed, its Lebesgue measure is zero and the condition

$$\sum_{\vee} |l_\vee|^{(1-2\alpha)/(1-\alpha)} < \infty$$

holds where $|l_\vee|$ is the length of the complementary intervals $l_\vee$ of the set S_2^+.

Proof. If q_i has the property (3.9) we know that from the systems of (2.4), (2.5)

$$|H_{i,j}(x, s)| \leq C\ \exp\Big[-\delta\Big(\frac{x+s}{2}\Big)^\alpha\Big], \qquad \delta > 0$$

and then

$$B_k = \int_0^\infty t^k |H_{i,j}(0,t)| dt \le A_1 d_1^k (1+k)^{(k+1)/\alpha - 1} \tag{3.10}$$

is satisfied, where A_1 and d_1, are constants depending on δ and α. By using

$$\left(1+\frac{1}{k}\right)^{k/\alpha} < e^{1/\alpha}, \quad (1+k)^{1/\alpha-1} < e^{(k+1)/\alpha}, \quad k^k < k!, e^k$$

and (3.10) we obtain that

$$B_k \le A d^k k! k^{k(1-\alpha)/\alpha} \tag{3.11}$$

is satisfied, where A and d are constants depending on δ and α. From (3.3) we have

$$|D^{(k)}(\lambda)| \le C d^k k! k^{k(1-\alpha)/\alpha} \tag{3.12}$$

(3.12) gives us that $D_+(\lambda) \in G_{\alpha/(1-\alpha)}$ and $S_2^+ \notin E_{\alpha/(1-\alpha)}$, where $G_{\alpha/(1-\alpha)}$ shows the Gevrey class of the order $\alpha/(1-\alpha)$ and $E_{\alpha/(1-\alpha)}$ shows the class of uniqueness set (see [3]). Then using Carleson's theorem [1] we obtain that S_2^+ is closed, its Lebesgue measure zero and the following inequality is satisfied

$$\sum_{\vee} |l_\vee|^{1-\alpha/(1-\alpha)} < \infty.$$

Thus the proof of theorem 3.4 is completed.

We showed that $D_+(\lambda) \in G_{\alpha/(1-\alpha)}$ and $S_2^+ \notin E_{\alpha/(1-\alpha)}$. For the structure of S_2^+ see also Hruscev's theorem given in [3].

THEOREM 3.5. *If the functions $q_i(x)$, $i = 1,2$, satisfy the inequalities of*

$$|q_i(x)| \le C \exp(-\delta\sqrt{x}), \quad i = 1,2, \quad \delta > 0 \tag{3.13}$$

then the set S_2^+ is empty.

Proof. By using the hypothesis (3.13) and the relation (3.11) we can obtain the following result

$$B_k \le A.d^k.k^{2k}.$$

Comparing the relation (3.13) and the hypothesis of theorem 3.3 we can see that under (3.13) the theorem 3.3 holds good. Using $k^k \leq k!e^k$ we arrive the result as follows

$$(3.14) \quad T(s) \leq \inf_k \frac{cd^k s^k k^{2k}}{k!} \leq \inf_k (d^k s^k e^k k^k) \leq C \exp(-d^{-1}e^{-1}s^{-1})$$

Getting help from (3.8) and (3.14) we see that

$$(3.15) \qquad \int_0^{l_0} \frac{d\theta(S_2^+, s)}{S} < \infty$$

is satisfied, where l_0 is the length of the longest bounded complementary intervals of S_2^+. The condition (3.15) is satisfied if and only if the linear measure $\theta(S_2^+, s) = 0$, for all s. This explains that S_2^+ must be an empty set. Thus we have completed the proof of the theorem.

According to the theorem 3.3 $S_3^+ \subset S_2^+$ it is easy to see that under the condition (3.13) the set S_3^+ is empty.

COROLLARY 3.1. *When the inequality (3.13) is satisfied then the eigen values and spectral singularities of the operator L is finite. This is also true for the multiplicity of them.*

When the condition (3.13) holds the spectral expansion formula related with the eigen functions of the operator L will be examined in a different work.

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Ankara University, Faculty of Sciences
Department of Mathematics
Ankara-TURKEY