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AN EXTENSION OF THE HONG–PARK VERSION OF THE CHOW–ROBBINS THEOREM ON SUMS OF NONINTEGRABLE RANDOM VARIABLES

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A famous result of Chow and Robbins [8] asserts that if $\{X_n, n \geq 1\}$ are independent and identically distributed (i.i.d.) random variables with $E|X_1| = \infty$, then for each sequence of constants $\{M_n, n \geq 1\}$ either

$$\liminf_{n\to\infty} \left| \frac{\sum_{j=1}^{n} X_j}{M_n} \right| = 0 \text{ almost certainly (a.c.)} \tag{1}$$

or

$$\limsup_{n\to\infty} \left| \frac{\sum_{j=1}^{n} X_j}{M_n} \right| = \infty \quad \text{a.c.} \tag{2}$$

and thus $P\{\lim_{n\to\infty} \sum_{j=1}^{n} X_j / M_n = 1\} = 0$. Note that both (1) and (2) may indeed prevail.

Aaronson [1] proved that the Chow-Robbins theorem holds if the i.i.d. hypothesis is replaced by the assumption that $\{X_n, n \geq 1\}$ is an ergodic, nonnegative strictly stationary sequence. Aaronson [2] showed that the nonnegativity proviso may be dispensed with if $\limsup_{n\to\infty} |M_n|/n = \infty$. Recently, Hong and Park [13] proved that the Chow-Robbins theorem holds if the$\{X_n, n \geq 1\}$ are assumed to be merely pairwise i.i.d. (p.i.i.d.). The Hong-Park theorem provides a welcome complement to a result of Etemadi [11] asserting that the classical Kolmogorov strong law of large numbers (SLLN) for i.i.d. integrable random variables holds if the random variables are p.i.i.d. It should be

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mentioned that the central limit theorem (CLT) can fail in the p.i.i.d. case. For some examples and discussion about the failure of the CLT, see Bradley [7], Cuesta and Matrán[10], Janson [14], and Romano and Siegel [15, Example 5.45].

Adler and Rosalsky [5] generalized the Chow-Robbins theorem to the case of weighted i.i.d. random variables by the following theorem. Adler [3] showed that a similar result holds for weighted sums of i.i.d. multidimensionally indexed random variables.

THEOREM 1. (Adler and Rosalsky [5]) *Let* $S_n = \sum_{j=1}^n a_j Y_j$, $n \geq 1$, *where* $\{a_n, n \geq 1\}$ *are nonzero constants and* $\{Y_n, n \geq 1\}$ *are i.i.d. random variables. If* $E|Y_1| = \infty$, $n|a_n| \uparrow$, *and* $\sum_{j=1}^n |a_j| = O(n|a_n|)$, *then for each sequence of constants* $\{M_n, n \geq 1\}$ *either*

$$\liminf_{n\to\infty} \left|\frac{S_n}{M_n}\right| = 0 \quad \text{a.c. or} \quad \limsup_{n\to\infty} \left|\frac{S_n}{M_n}\right| = \infty \quad \text{a.c.}$$

and, consequently, $P\{\lim_{n\to\infty} S_n/M_n = 1\} = 0$.

The purpose of the current work is to generalize Theorem 1 in two directions, namely (i) the summands are random elements in a real separable Banach space, and (ii) the hypothesis of "i.i.d." is replaced by "p.i.i.d."This will be accomplished by the following theorem which is thus an extension of the Hong-Park [13] version of the Chow-Robbins [8] theorem.

THEOREM 2. *Let* $S_n = \sum_{j=1}^n a_j V_j$, $n \geq 1$, *where* $\{a_n, n \geq 1\}$ *are nonzero constants and* $\{V_n, n \geq 1\}$ *are p.i.i.d. random elements in a real separable Banach space. If*

$$E||V_1|| = \infty, \tag{3}$$

$$n|a_n| \uparrow, \tag{4}$$

and

$$\sum_{j=1}^n |a_j| = O(n|a_n|), \tag{5}$$

then for each sequence of constants $\{M_n, n \geq 1\}$ *either*

$$(6) \qquad \liminf_{n\to\infty} \left\|\frac{S_n}{M_n}\right\| = 0 \quad \text{a.c. or} \quad \limsup_{n\to\infty}\left\|\frac{S_n}{M_n}\right\| = \infty \quad \text{a.c.}$$

The authors take great pleasure to acknowledge that their proof of Theorem 2 owes much to the work of Hong and Park [13] in that versions of their two key lemmata are obtained below for weighted sums of p.i.i.d. random elements. Moreover, the argument establishing Theorem 2 is structually similar to that of the Hong-Park result. The main difference in the proofs of Theorems 1 and 2 is that the proof of Theorem 2 employs Lemma 2 whereas that of Theorem 1 employs a weighted i.i.d. version of a generalized SLLN of Feller [12] for i.i.d. random variables.

LEMMA 1. *Let* $S_n = \sum_{j=1}^{n} a_j V_j$, $n \geq 1$, *where* $\{a_n, n \geq 1\}$ *are nonzero constants and* $\{V_n, n \geq 1\}$ *are p.i.i.d. random elements in a real separable Banach space, and let* $\{b_n, n \geq 1\}$ *be positive constants satisfying*

$$b_n = O(b_{n+1}) \quad \text{and} \quad \frac{b_n}{n^{\alpha}|a_n|} \uparrow \quad \text{for some } \alpha > 0.$$

If

$$(7) \qquad \sum_{n=1}^{\infty} P\{\|a_n V_1\| > b_n\} = \infty,$$

then

$$(8) \qquad \limsup_{n\to\infty} \frac{\|a_n V_n\|}{b_n} = \limsup_{n\to\infty} \frac{\|S_n\|}{b_n} = \infty \quad \text{a.c.}$$

Proof. The proof is identical to that of Lemma 1(ii) of Adler and Rosalsky [5] except that instead it uses the "pairwise independent" version of Borel-Cantelli lemma (see, e.g., Chung [9, Theorem 4.2.5]).

As in Lemma 2 of Hong and Park [13], pairwise independence is also not needed in the ensuing lemma. Lemma 2 will be stated in a form which is more general than that which is required for the current work and may be of independent interest. The only condition on V of a moment nature is (10).

LEMMA 2. *Let $S_n = \sum_{j=1}^{n} a_j V_j$, $n \geq 1$, where $\{a_n, n \geq 1\}$ are nonzero constants and $\{V_n, n \geq 1\}$ are random elements in a real separable Banach space, and let $\{b_n, n \geq 1\}$ be positive constants satisfying*

$$b_n \uparrow \infty, \quad \frac{b_n}{n|a_n|} \to \infty, \quad \frac{b_n}{n|a_n|} = O\left(\inf_{j \geq n} \frac{b_j}{j|a_j|}\right), \quad \text{and}$$

$$\sum_{j=1}^{n} |a_j| = O(n|a_n|).$$

Suppose that $\{V_n, n \geq 1\}$ is stochastically dominated by a random element V in the sense that for some constant $D < \infty$,

$$P\{||V_n|| > t\} \leq D\, P\{||DV|| > t\}, \ t \geq 0, \ n \geq 1. \tag{9}$$

If

$$\sum_{n=1}^{\infty} P\{||a_n V|| > Db_n|\} < \infty, \tag{10}$$

then for every subsequence $\{b_{n(k)}, k \geq 1\}$

$$\liminf_{k \to \infty} \frac{||S_{n(k)}||}{b_{n(k)}} = 0 \quad \text{a.c.}$$

irrespective of the joint distributions ofthe random elements $\{V_n, n \geq 1\}$.

Proof. Let $c_n = b_n/|a_n|$, $n \geq 1$, and $W_n = V_n I(||V_n|| \leq D^2 c_n)$, $n \geq 1$. Observe that(9) and (10) ensure that

$$\sum_{n=1}^{\infty} P\{a_n V_n \neq a_n W_n\} = \sum_{n=1}^{\infty} P\{||V_n|| > D^2 c_n\}$$

$$\leq D \sum_{n=1}^{\infty} P\{||V|| > Dc_n\} < \infty$$

and then by the Borel-Cantelli lemma

$$P\{a_n V_n \neq a_n W_n \quad \text{i.o.(n)}\} = 0.$$

Thus, it suffices to show that for every subsequence $\{b_{n(k)}, k \geq 1\}$ that

$$\liminf_{k\to\infty} \frac{\|\sum_{j=1}^{n(k)} a_j W_j\|}{b_{n(k)}} = 0 \text{ a.c.} \tag{11}$$

Now

$$\lim_{n\to\infty} b_n^{-1} \sum_{j=1}^{n} |a_j| E\|V_j\| I(\|V_j\| \leq D^2 c_j) = 0$$

as was shown in the *proof* of Theorem 6 of Adler, Rosalsky, and Taylor [6] whence

$$\begin{aligned} &E\left\{\liminf_{k\to\infty} \frac{\|\sum_{j=1}^{n(k)} a_j W_j\|}{b_{n(k)}}\right\} \\ &\leq \liminf_{k\to\infty} E\left\{\frac{\|\sum_{j=1}^{n(k)} a_j W_j\|}{b_{n(k)}}\right\} \quad \text{(by Fatou's lemma)} \\ &\leq \liminf_{k\to\infty} b_{n(k)}^{-1} \sum_{j=1}^{n(k)} |a_j| E\|V_j\| I(\|V_j\| \leq D^2 c_j) \\ &= 0 \end{aligned}$$

implying (11).

Proof of Theorem 2. Assume that the conclusion (6) fails. Then there are constants $\{M_n, n \geq 1\}$ such that

$$P\left\{\liminf_{n\to\infty} \left\|\frac{S_n}{M_n}\right\| = 0\right\} < 1 \quad \text{and} \quad P\left\{\limsup_{n\to\infty} \left\|\frac{S_n}{M_n}\right\| = \infty\right\} < 1. \tag{12}$$

Firstly, suppose that $M_n = O(n|a_n|)$ and set $b_n = n|a_n|$, $n \geq 1$. Then by the second half of (12), the contrapositive of Lemma 1, and the argument employed by Adler and Rosalsky [5] in the proof of Theorem 1, it follows that $E\|V_1\| < \infty$ which contradicts the hypothesis (3).

Next, suppose that

$$\limsup_{n\to\infty} \left|\frac{M_n}{n a_n}\right| = \infty. \tag{13}$$

Adler and Rosalsky [5] showed in the *proof* of Theorem 1 that there is a subsequence $n(k) \uparrow \infty$ such that

$$(14) \qquad \max_{1 \le j \le n(k)} \left| \frac{M_j}{j a_j} \right| = \left| \frac{M_{n(k)}}{n(k) a_{n(k)}} \right|, \quad k \ge 1.$$

Set $b_n = n|a_n| \max_{1 \le j \le n} |M_j/(j a_j)|$, $n \ge 1$. Then $b_n \ge |M_n|$, $n \ge 1$, and hence by the second half of (12)

$$(15) \qquad P\left\{ \limsup_{n \to \infty} \frac{||S_n||}{b_n} = \infty \right\} < 1$$

so (8) does not hold. By (13), $b_n/(n|a_n|) \uparrow \infty$ and then combining this with (4) it follows that $b_n \uparrow \infty$. Thus (15) ensures via the contrapositive of Lemma 1 that the series of (7) converges. Then, recalling (5), by Lemma 2 with $D = 1$ and $V = V_1$

$$\liminf_{k \to \infty} \frac{||S_{n(k)}||}{b_{n(k)}} = 0 \quad \text{a.c.}$$

But this and (14) imply that

$$\liminf_{k \to \infty} \left|\left| \frac{S_{n(k)}}{M_{n(k)}} \right|\right| = 0 \quad \text{a.c.}$$

and so

$$\liminf_{n \to \infty} \left|\left| \frac{S_n}{M_n} \right|\right| = 0 \quad \text{a.c.}$$

which contradicts the first half of (12) thereby provingthe theorem.

The following example shows that Theorem 2 can fail if proviso (5) is dispensed with.

EXAMPLE. Consider the real separable Banach space l_1 of absolutely summable real sequences $s = \{s_i, i \ge 1\}$ with norm $||s|| = \sum_{i=1}^{\infty} |s_i|$ and let $s^{(k)}$ denote the element of l_1 having 1 in its k^{th} position and 0 elsewhere, $k \ge 1$. Let $\{K_n, n \ge 1\}$ be i.i.d. positive integer-valued random variables and let $\{Y_n, n \ge 1\}$ be i.i.d. random variables with the generalized St. Petersburg distribution

$$P\{Y_1 = q^{-y}\} = pq^{y-1}, \quad y = 1, 2, \ldots$$

where $0 < p = 1 - q < 1$. Furthermore, suppose that $\{K_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ are independent stochastic processes. Let

$$V_n = \sum_{k=1}^{\infty} Y_n s^{(k)} I(K_n = k),\ n \geq 1.$$

Then $\{V_n, n \geq 1\}$ are i.i.d. random elements in l_1 with $E||V_1|| = EY_1 = \infty$. If $a_n = n^{-1}$, $n \geq 1$, then (4) holds but (5) fails. Then letting log denote the logarithm to the base q^{-1},

$$\left|\left|\frac{\sum_{j=1}^{n} j^{-1} V_j}{p(2q\log e)^{-1}(\log n)^2}\right|\right| = \frac{\sum_{j=1}^{n} j^{-1} Y_j}{p(2q\log e)^{-1}(\log n)^2} \to 1 \quad \text{a.c.}$$

as was shown by Adler and Rosalsky [5], whence (6) fails.

REMARK. The topic of this paper can be given an interesting interpretation in connection with the theory of "fair games" and the curious reader may refer to the papers by Chow and Robbins [8], Adler and Rosalsky [5], and Adler [4] for a discussion of this and related topics.

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