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COMPLETE OPEN MANIFOLDS AND HOROFUNCTIONS

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1. Introduction

Let M be a complete open Riemannian manifold. When the sectional curvature K_M of M is nonpositive, Gromov has defined, in his lectures [3], the *ideal boundary* of M, and used it to study the geometric structure of M. In a Hadamard manifold, a simply connected manifold with nonpositive sectional curvature, a point at infinity can be defined as an equivalence class of rays. He proved many interesting theorems using this definition of ideal boundary and the so-called *Tits' metric* on it. He also suggested a counterpart to this for nonnegative curvature case. This idea has been taken up by Kasue to study the structure of complete open manifolds with asymptotically nonnegative curvature [14]. Motivated by these works, we will define an ideal boundary of a general noncompact manifold M, and study its structure.

There are two different directions when one tries to define the ideal boundary. We can define the ideal boundary as equivalence classes determined by an asymptotic relation of rays, or as the limits of distance functions in the set of all continuous functions on M. In the case of a Hadamard manifold, one can define an asymptotic relation so that these two methods in fact coincide. In general, however, the same kind of asymptotic relation does not behave as nicely as we want.

One of the main objectives of this paper is to show when we can carry out the construction of the ideal boundary of a complete open manifold in the same manner as a Hadamard manifold. This problem was studied in detail for the case of surfaces [16], and we will generalize some of the results to higher dimensional manifolds. A natural choice

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of a generalization of a manifold with nonpositive curvature might be a manifold without conjugate points. However, unfortunately, it is not even known whether the asymptotic relation is symmetric for a manifold without conjugate points, and we will need a little stronger condition, for example, a manifold without focal points.

A manifold without conjugate points has long been an object of study in connection with generalizations of results in manifolds with nonpositive curvature. Maybe the most important open question about manifolds without conjugate points is the following problem, which is often called the *Hopf-conjecture*: Any Riemannian n-torus without conjugate points is flat. E. Hopf solved this problem for the 2-torus [13]. The general case of the n-torus is easy to see under the stronger condition of nonpositive curvature, and was proved by Avez [1] under the assumption of no focal points. We will follow this line, and show that under certain conditions the ideal boundary of the universal covering space of a compact manifold without conjugate points has the same properties as the nonpositive curvature case. As examples of such manifolds we will have manifolds without focal points and manifolds with geodesic flow of Anosov type.

2. Ideal boundary of an open manifold

A unit speed geodesic $\gamma : [0, \infty) \to M$ is called a *ray* if $d(\gamma(s), \gamma(t)) = |s - t|$ for any s, $t \in [0, \infty)$. For any $p \in M$ let $\{q_i\}$ be a divergent sequence of points in M and for each i let γ_i be a unit speed minimal geodesic from p to q_i. Then the sequence $\{\gamma_i\}$ has a convergent subsequence and the limit is a ray emanating from p. Therefore an open manifold has rays emanating from every point. A ray $\sigma : [0, \infty) \to M$ is said to be *asymptotic* to a ray $\gamma : [0, \infty) \to M$ if there exist a monotone sequence $\{t_n\}$ of positive numbers such that $t_n \to \infty$ as $n \to \infty$ and a sequence $\{\sigma_n : [0, \ell_n] \to M\}$ of minimal geodesics such that $\sigma_n(\ell_n) = \gamma(t_n)$ for each n and $\sigma_n \to \sigma$ as $n \to \infty$. An asymptotic ray emanating from a fixed point is not unique in general, and the asymptotic relation is not symmetric, which means γ may not be asymptotic to σ (see [4], [16]).

In the case of a Hadamard manifold, however, this asymptotic relation becomes an equivalence relation, and in fact two unit speed geodesics $\gamma_i : \mathbb{R} \to M$ are equivalent if there is a constant $a \in \mathbb{R}$, such that $d(\gamma_1(t), \gamma_2(t)) \le a$ for all $t \ge 0$. The equivalence classes of this relation are called the *points at infinity* and are denoted by $M(\infty)$. Since we can not expect this beautiful property in a general noncompact manifold, we need a more universal way of compactification. For this purpose we will follow the lead of [3], [12].

Let M be a non-compact complete Riemannian manifold with distance function $d(\ ,\ )$. For each $x \in M$ let d_x be the continuous function $y \to d(x,y)$. The map $x \to d_x$ defines an embedding of M into $C(M)$, the set of continuous functions on M with the topology of uniform convergence on compact sets. We consider $C_*(M) \overset{\text{def}}{=} C(M)/(\text{constant functions})$ with the induced topology and the induced embedding $i : M \to C_*(M)$ defined by $i(x) = \bar{d}_x$, the equivalence class of d_x. The boundary ∂M is defined as $Cl(M) - i(M)$, where $Cl(M)$ is the closure of $i(M)$ in $C_*(M)$. A point in ∂M is an equivalence class of functions called *horofunctions*, which are well-defined up to a constant.

For a given ray $\gamma : [0, \infty) \to M$ and a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$, the limit horofunction of the distance function $d_{\gamma(t_n)}$ is called a *Busemann-function.* The Busemann-function b_γ of γ can be written as

$$b_\gamma(x) = \lim_{t \to \infty} (d(x, \gamma(t)) - t).$$

In a sense, the Busemann-function b_γ is a distance function of a point at infinity which is at the end of the ray γ. For a Hadamard manifold, it is known that all horofunctions are Busemann-functions and two rays define the same Busemann-function if and only if they are asymptotic. Therefore the ideal boundary can be defined by the set of Busemann-functions or the equivalence classes of rays representing the same Busemann functions up to constants, that is, $M(\infty) = \partial M$.

Let M be an open manifold and $Cl(M) \subset C_*(M)$ be the compactification of M by horofunctions. Maybe the most significant difference between $Cl(M)$ and a compact domain with boundary is the following. Every boundary point of a compact domain can be connected to an interior point by a minimal geodesic. Suppose a horofunction $h \in \partial M$ is connected to an interior point $p \in M$ by a ray $\gamma : [0, \infty) \to M$. Then h is the limit of the equivalence class $\bar{d}_{\gamma(t)}$. Therefore, in fact, we are looking for a ray defining a Busemann-function which agrees with h up to a constant. It is not always possible to find such a ray in general, and if this can be done, then of course every horofunction must be a Busemann-function. As in [16], we use the following concept to describe this property.

DEFINITION 2.1. A complete open manifold M is said to have a *convex* ideal boundary at infinity if for any $p \in M$ and any horofunction $h \in \partial M$ there is a ray $\gamma : [0, \infty) \to M$ such that $\gamma(0) = p$ and the Busemann-function b_γ agrees with h up to a constant.

In a sense, we may say this definition is a generalization of a bounded convex domain. Since M is complete, any two interior points can be connected by a minimal geodesic by the Hopf-Rinow theorem. A ray plays the role of a minimal connection between an interior point and a boundary point at infinity.

If M has a convex ideal boundary, then for any $p \in M$ every point in ∂M can be represented by a Busemann-function determined by a ray emanating from p. Therefore the ideal boundary can be defined either by equivalence classes of rays, or the set of horofunctions. In the next section, we will find a sufficient condition for an open manifold to have a convex ideal boundary.

3. Open manifolds with convex boundary

Let M be a complete, simply connected Riemannian manifold without conjugate points, and let SM be the unit tangent bundle of M. For each unit vector $v \in S_pM$ let $\gamma_v = \exp_p tv$ be the unit speed geodesic. Since M has no conjugate points, it is easy to see that γ is a ray emanating from the base point p of v and defines a Busemann-function, which we will denote by b_v. If b_v is normalized so that $b_v(p) = 0$, we call $H_v = b_v^{-1}(0)$ the *horosphere* and $B_v = b_v^{-1}((0, \infty))$ the *horoball* of v. A vector $w \in S_qM$ for an arbitrary $q \in M$ is called *asymptotic* to v if the ray γ_w is asymptotic to γ_v. We then have the following results [8], [15].

PROPOSITION 3.1. *Let M be a complete, simply connected Riemannian manifold without conjugate points. Then for each $v \in SM$ the Busemann-function b_v is C^1-differentiable, and the gradient $\nabla b_v(q)$*

of b_v at $q \in M$ is a unit vector asymptotic to v at q. In particular, an asymptotic vector is unique at each point of M.

The behavior of a Busemann-function b_v is closely related to the geometric structure of the level set H_v, more precisely, the shape operator of the horosphere. We will use the Lagrange tensors to investigate the geometry of the horosphere and hence the Busemann-function itself.

Let $\gamma_v : [0, \infty) \to M$ be a ray in M and $R_v = R_v(t)$ be the tensor field along γ_v defined by $R_v(t)V = R(V, \gamma_v'(t))\gamma_v'(t)$ for any vector field $V(t)$ orthogonal to $\gamma_v'(t)$, where $R(\cdot, \cdot)$ is the curvature tensor of M. Then a matrix solution to the differential equation,

$$Z'' + R_v Z = 0 \ ,$$

is called a *Jacobi tensor.* Each Jacobi tensor, applied to a parallel normal vector field along γ_v, gives rise to an $(n-1)$-dimensional space of Jacobi fields along γ_v when the dimension of M is n. A Jacobi tensor $Z(t)$ is called a *Lagrange tensor* if the Wronskian $W(Z, Z) = Z'Z^* - Z^*Z'$ vanishes, where $*$ denotes the adjoint with respect to the Riemannian metric. If Z is nowhere singular, this is equivalent to saying that the tensor $Z'Z^{-1}$ and Z^*Z' are symmetric.

The importance of the Lagrange tensors consists in the following geometric description: Let H be an oriented C^2-hypersurface in M. The normal bundle has a canonical trivialization using the oriented unit normal vectors: $NH = H \times \mathbb{R}$. There exists a neighborhood U of the zero-section such that the mapping $\exp|_U : U \to M$ is a diffeomorphism. Let V be the unit vector field over U defined by the velocity vectors of the geodesics normal to H. Consider a V-invariant vector field J along a fixed normal geodesic γ. Then of course $J(t)$ is a Jacobi field along γ, and it is easy to see it satisfies

$$J' = (\nabla V)J,$$

where ∇V is the tensor field $v \to \nabla_v V$ for all v orthogonal to γ. We hence take a tensor equation

$$Z' = (\nabla V)Z$$

for some normal tensor field $Z(t)$ along γ which is uniquely determined by its initial value $Z(0)$. A matrix solution $Z(t)$ is nonsingular along a geodesic in U, and $-Z'Z^{-1}(0) = -\nabla V(\gamma(0))$ is the shape operator of H at $\gamma(0)$ which is clearly symmetric. Therefore we see that Z is a Lagrange tensor. In fact, it can be shown that each Lagrange tensor along any geodesic arises in this way, and we will say that Z is *related* to H if $Z'Z^{-1}(t)$ is the shape operator of H for some t.

We now return to the geometry of the horosphere H_v defined by a unit vector $v \in SM$. For each point $q \in H_v$ a unit vector $w \in S_qM$ asymptotic to v is the gradient of b_v, and hence orthogonal to H_v. It is therefore clear that the asymptotic vectors give rise to a canonical trivialization of the normal bundle NH_v, and one might want to apply the argument above. In order to apply the above geometric description of the Lagrange tensors to the hypersurface H_v, we need C^2-differentiability, which is unfortunately not guaranteed by Proposition 3.1. For this reason, we will need some extra conditions to properly understand the Busemann-functions.

For a geodesic $\gamma : \mathbb{R} \to M$, let D_s be the Lagrange tensor along γ satisfying the boundary condition

$$D_s(0) = Id, \; D_s(s) = 0.$$

Since M has no conjugate points, we know that D_s uniquely exists for each $s \in \mathbb{R}$, and nonsingular for all $t \neq s$. It is in fact related to the metric sphere centered at $\gamma(s)$, which is clearly C^∞ because there are no conjugate points in M. From the theory of differential equations it follows that as $s \to \infty$ the field D_s converges to some Lagrange tensor D called the *stable Jacobi tensor* along γ, if and only if $\lim_{s\to\infty} D_s'(0)$ exists. It is well known [9], [11] that for manifolds without conjugate points this limit always exists, and we have $D(0) = Id$ and $D'(0) = \lim_{s\to\infty} D_s'(0)$. Since the distance function $d_{\gamma(s)}$ converges to the Busemann-function b_γ as $s \to \infty$ in the topology of uniform convergence on compact sets, the metric sphere with center at $\gamma(s)$ will converge to the horosphere $H_{\gamma'(0)}$ as $s \to \infty$ on each compact subset of M. Therefore we may regard D as a Lagrange tensor related to the horosphere even though it is only C^1-differentiable.

In the study of manifolds without conjugate points, the stable tensor field has played a central role. To see how, we first need the following concept: A Jacobi field Z_v defined for each $v \in SM$ is called *continuous* if the initial values $Z_v(0)$ and $Z_v'(0)$ are continuous as tensors of the vector bundle $\{(x,v) \in TM \times SM |\ x \perp v\}$ over SM. For each unit vector $v \in SM$ let D_v be the stable Jacobi tensor defined by the geodesic γ_v. In [13], Hopf remarks that D'_v (for a surface it is the geodesic curvature at $\gamma_v(0)$ of the metric circle with center at $\gamma_v(s)$) is continuous in v. It turns out this is not true in general and a counterexample is constructed in [2]. The continuity of the stable Jacobi tensor is an essential tool as we can see in the following proposition (see [8] for the proof).

PROPOSITION 3.2. *Let M be a complete, simply connected manifold without conjugate points. If the stable Jacobi tensor D_v is continuous, then the convergence of the gradient ∇d_s to ∇b_v is uniform on each compact subset of M, where $d_s(q) = d(\gamma_v(s), q)$ for each $q \in M$. In particular, b_v is $C^{1,\alpha}$-differentiable for any $0 < \alpha < 1$.*

The proof of this proposition essentially depends on the fact that for each compact subset $K \subset M$ and a fixed number $D > 0$, if $x \in M$ is a point such that $d(x, K) \geq D$, the gradient ∇d_x of the distance function d_x is a Lipschitz function on K with a Lipschitz constant depending only on K and D. We can apply the same idea to a general horofunction.

Let $h \in \partial M$ be a horofunction. Then there exists a sequence $\{x_n\}$ of points in M such that $d(x_n, p) \to \infty$ as $n \to \infty$ for each $p \in M$ and the equivalence class $\bar{d}_{x_n}$ converges to h in $C_*(M)$. For a fixed point $p \in M$ put $d_n(q) = d(x_n, q) - d(x_n, p)$. Then on each compact subset of M the sequence d_n uniformly converges to a horofunction h such that $h(p) = d_n(p) = 0$. We can then prove the following by the same argument as the above proposition.

LEMMA 3.3. *Let M be a complete, simply connected manifold without conjugate points. If the stable Jacobi tensor is continuous, then on each compact subset of M the sequence $\{\nabla d_n\}$ has a uniformly convergent subsequence.*

Proof. For each compact subset $K \subset M$ and a fixed number $D > 0$ we can choose N so that for each $n \geq N$, ∇d_n is Lipschitz continuous with a fixed constant L on K. Therefore $\{\nabla d_n |\ n \geq N\}$ is equicontinuous. Since $||\nabla d_x|| \leq 1$ for any distance function, we can apply Ascoli's theorem to see there exists a unformly convergent subsequence of $\{d_n\}$. □

Without loss of generality we will assume that the sequence $\{d_n\}$ uniformly converges when a compact subset is given. By the lemma, it is now easy to see that the horofunction h is in fact $C^{1,\alpha}$-differentiable.

There are several cases when we can obtain stable Jacobi fields [8], [9]. One case of particular importance is the following: A complete manifold M is called a *manifold with bounded asymptote* if it is connected without conjugate points, and if there exists a uniform bound $B \geq 1$ for stable Jacobi tensor D such that for all $v \in SM$ and $t \geq 0$

$$||D_v(t)|| \leq B,$$

where the norm $||\cdot||$ for a tensor T is given by $||T|| = \max\{||T(v)||;\ ||v|| = 1\}$. In fact, for such manifolds, if we denote by D_{vs} the Lagrange tensor along γ_v such that $D_{vs}(0) = Id$ and $D_{vs}(s) = 0$, then the convergence of $D'_{vs}(0) \to D'_v(0)$ is uniform for all v in any compact subset of SM and hence the limit is continuous with respect to v [8]. For manifolds without focal points $||D_v(t)||$ is monotonely decreasing. Since $D_v(0) = Id$ for any $v \in SM$, they have bounded asymptote. Another class of examples are manifolds with geodesic flow of Anosov type [6], [7].

For each horofunction h given as a limit of a sequence $\{d_n\}$ and any fixed point $q \in M$ consider the set of minimal geodesics $\{\sigma_n : [0, l_n] \to M\}$ such that $\sigma_n(0) = q$ and $\sigma_n(l_n) = x_n$. As $n \to \infty$, σ_n converges to a ray which we denote by σ_q, and it is easy to see [15], [16] that $\sigma'_q(s) = -\nabla h(\sigma_q(s))$. Since h is C^1-differentiable, for each $q \in M$ there exists a unique such ray σ_q, and we call σ_q and its velocity vectors asymptotic to h. In order to prove our main theorem we need the following:

LEMMA 3.4. *If M is a complete, simply connected manifold with bounded asymptote, then for any two rays σ_p and σ_q asymptotic to a horofunction h there exists a constant $a > 0$ such that $d(\sigma_p(t), \sigma_q(t)) \leq a$ for all $t \in [0, \infty)$.*

Proof. Without loss of generality we assume that $h(p) = h(q) = 0$, and h is a limit of d_n with $d_n(p) = 0$. Therefore, for each $t \in [0, \infty)$, $\sigma_p(t)$ and $\sigma_q(t)$ are on the same level set of h. Since σ_p and σ_q are asymptotic to h, for any fixed number $T \in [0, \infty)$ and any $\varepsilon > 0$ there exists a number $N_1 > 0$ such that for all $n \geq N$ we have

$d(\sigma_p(T), \gamma_{pn}(T)) < \varepsilon$, $d(\sigma_q(T), \gamma_{qn}(T)) < \varepsilon$, and $|d_n(q)| < \varepsilon$, where γ_{pn} and γ_{qn} are the minimal geodesics connecting p and q respectively to x_n from the remark proceeding the lemma. So it is enough to show that the distance $d(\gamma_{pn}(T), \gamma_{qn}(d_n(q) + T))$ is bounded by a fixed number $a > 0$ for a sufficiently large n.

Since $d_n(\gamma_{qn}(d_n(q))) = 0$ for each n we can find a C^1 curve $c_{n0} : [0, a_n] \to M$ connecting p and $\gamma_{qn}(d_n(q))$ such that $\|c'_{n0}\| = 1$ and $d_n(c_{n0}(r)) = 0$ for each $r \in [0, a_n]$, and the arclength of c_{n0} is bounded by a fixed number Q for all $n \geq N$. Denote by c_{nt} the image of c_{n0} by the flow of ∇d_n. Then we have $d_n(c_{nt}(r)) = t$ for all $r \in [0, a_n]$, and there exists a compact set $K \subset M$ which contains $\{c_{nt}(r) |\, (t, r) \in [0, T] \times [0, a_n]\}$ for all $n \geq N$.

Since M has bounded asymptote and $SK \subset SM$ is compact, there exist two positive numbers $L > 0$ and $n \geq N$ such that for any $(t, r) \in [0, T] \times [0, a_n]$ we have $\|D_{vs}(t)\| \leq L$, where $v = -\nabla d_n(c(r))$ and $s = l_n = d(p, x_n)$. For each $r \in [0, a_n]$ we denote by γ_r the geodesic $t \to c_{nt}(r)$. Then by the variation through these geodesics we obtain a Jacobi field J_r, which can be also expressed as $D_{vs}(V_r)$ for some parallel unit normal vector field V_r along γ_r because $\|J_r(0) = c'_{n0}(r)\| = 1$ and $\gamma'_r(0) \perp c'(r)$ for each r.

Since $\|J_r(t) = D_{vs}(V_r(t))\| \leq L$ for each $(t, r) \in [0, T] \times [0, a_n]$, by integrating along each c_{nt} we see that the arclength of c_{nt} is bounded by QL, and therefore the distance between $c_{nT}(0) = \gamma_{pn}(T)$ and $c_{nT}(a_n) = \gamma_{qn}(d_n(q) + T)$ is also bounded by the same constant $a = QL$. □

In connection with the asymptotic behavior of rays, Green introduced the following concept [10]: The geodesic rays emanating from $p \in M$ are said to *diverge uniformly* if for any two rays γ_1 and γ_2 from $p \in M$ we have $\lim_{t\to\infty} d(\gamma_1(t), \gamma_2(t)) = \infty$. Then the following lemma can be proved [9].

LEMMA 3.5. *Let $p \in M$ be a pole of a complete. connected, simply connected manifold M whose sectional curvatures are bounded from below. If the stable Jacobi tensors D_v exist for each $v \in S_pM$ and depend continuously on v then the geodesic rays going out from p diverge uniformly.*

This lemma is the final ingredient needed for our main theorem.

THEOREM 3.6. *Let M be the universal covering space of a compact manifold without conjugate points. If M has bounded asymptote, then M has a convex ideal boundary at infinity.*

Proof. Since M is the universal covering space of a compact manifold, it has curvatures bounded below. Furthermore, since M has no conjugate points and has bounded asymptote, each point in M is a pole and by Lemma 3.5 we see that the rays in M diverge uniformly.

For any horofunction $h \in \partial M$ and $p, q \in M$ let σ_p, σ_q be the rays asymptotic to h at p and q respectively. We claim that σ_p and σ_q are asymptotic to each other. Suppose there is a ray γ_q different from σ_q and asymptotic to σ_p. Then by Lemma 3.4 there exists a constant $b > 0$ such that $d(\sigma_p(t), \gamma_q(t)) < b$ for all $t \in [0, \infty)$. On the other hand by Lemma 3.5 we have $d(\sigma_q(t), \gamma_q(t)) \to \infty$ as $t \to \infty$, which is a contradiction because $d(\sigma_p(t), \sigma_q(t)) < a$ for some constant $a > 0$.

For a fixed point p with $h(p) = 0$ and any $q \in M$ the unique ray asymptotic to h is also asymptotic to σ_p. Since the asymptotic vector to a horofunction is same as its gradient vector, we conclude that the horofunction h and the Busemann-function b_{σ_p} has the same gradient vector fields. We already know that h and b_{σ_p} are C^1-differentiable and $h(p) = b_{\sigma_q}(p) = 0$. Therefore the two functions must coincide.

Now for any $p \in M$ we apply the same argument and conclude that there exists a ray emanating from p and the Busemann-function defined by it agrees with h upto a constant, and hence M has a convex ideal boundary. □

In the above proof, we in fact proved that the asymptotic relation is an equivalence relation on M. In [8], the author claimed the same conclusion under a weaker condition. It is not clear whether this fact is still true under this weaker condition, but the proof given seems incomplete as it is. It would be interesting to know how much we could relax the required conditions for Theorem 3.6. At the moment we are asking very much even though they still include important cases such as manifolds without focal points and manifolds with geodesic flow of Anosov type.

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