J. Korean Math. Soc. **32** (1995), No. 2, pp. 329–339

L_q ESTIMATION ON THE LEAST ENERGY SOLUTIONS

DAE HYEON PAHK AND SANG DON PARK

1. Introduction

Let us consider the Neumann problem for a quasilinear equation

$$(\mathrm{I}_\varepsilon) \qquad \begin{cases} \varepsilon^m div(|\nabla u|^{m-2}\nabla u) - u|u|^{m-2} + f(u) = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

where $1 < m < N$, $N \geq 2$, $\varepsilon > 0$, Ω is a smooth bounded domain in $\mathbf{R}^N$ and ν is the unit outer normal vector to $\partial\Omega$. Moreover f is in $C^2(\mathbf{R})$ and satisfies the following conditions:

(f_1) $f(t) = 0$ for $t \leq 0$
(f_2) $f(t)/t^{m-1} \uparrow +\infty$ as $t \to +\infty$
(f_3) $f(t) = o(t^{m-1})$ at 0^+ and $f(t) = O(t^{p-1})$ at $+\infty$, where $m < p < \frac{mN}{N-m}$
(f_4) there exists $\theta \in (0, \frac{1}{m})$ such that $F(t) = \int_0^t f(s)\,ds \leq \theta t f(t)$

In this paper, we study the asymptotic L_q behavior of the mountain pass solutions, as $\varepsilon \to 0$, which are critical points of the functional J_ε defined on $W^{1,m}(\Omega)$ by

$$(1.1) \qquad J_\varepsilon(v) = \frac{1}{m}\int_\Omega \varepsilon^m|\nabla v|^m + |v|^m\,dx - \int_\Omega F(v)\,dx$$

In relation to J_ε, a critical value is determined by

$$(1.2) \qquad c_\varepsilon = \inf_{g\in\Gamma}\sup_{0\leq t\leq 1} J_\varepsilon(g(t)),$$

Received March 21, 1994.
1991 AMS Subject Classification: 35A15.
Key words and phrases: least energy solution, mountain pass method, sobolev imbedding.

where $\Gamma = \{ g \in C([0,1], W^{1,m}(\Omega)) \mid g(0) = 0 \, \text{and} \, g(1) = e \}$ and e is a nontrivial and nonnegative element in $W^{1,m}(\Omega)$ such that $J_\varepsilon(e) \le 0$.

In the case $m = 2$, this model is reduced to a semilinear problem which is deeply studied by C.H. LIN, W.M.-NI and I. TAKAGI,[6, 7]. They also showed that the peak point of the mountain pass solution corresponding to sufficiently small ε is located at the boundary point whose mean curvature is maximal. We discuss the existence of the mountain pass solution of (I_ε) in section 2 and estimate the growth of internal energy

$$E_\varepsilon(v) = \int_\Omega \varepsilon^m |\nabla v|^m + |v|^m \, dx.$$

and show the L_q-behavior of the solutions in terms of ε in section 3.

2. Existence of positive solution of (I_ε)

We first verify the constant c_ε is a positive critical value of J_ε, whose corresponding critical point is called a mountain pass solution or a least energy solution.

LEMMA 2.1. *c_ε in (1.2) is a positive critical value of J_ε defined in (1.1)*

Proof. Since J_ε is C^1 functional, it suffices to show that J_ε satisfies all hypotheses of Mountain Pass lemma, [8, Theorem 2.2]. Let $\{u_j\} \subset W^{1,m}(\Omega)$ be a Palais Smale sequence, that is, there is a positive constant a_1 satisfying $|J_\varepsilon(u_j)| < a_1$ and $DJ_\varepsilon(u_j) \to 0$. Then we obtain by (f_4),

$$\begin{aligned} a_1 + o(1)\|u_j\|_{W^{1,m}} &\ge J_\varepsilon(u_j) - \theta \langle DJ_\varepsilon(u_j), u_j \rangle \\ &= (\frac{1}{m} - \theta) \int_\Omega \varepsilon^m |\nabla u_j|^m + |u_j|^m \, dx \\ &\quad + \int_\Omega (\theta u_j f(u_j) - F(u_j)) \, dx \\ &\ge (\frac{1}{m} - \theta) \int_\Omega \varepsilon^m |\nabla u_j|^m + |u_j|^m \, dx \end{aligned}$$

Hence $\{u_j\}$ is bounded in $W^{1,m}(\Omega)$. So there is a subsequence, again denoted by $\{u_j\}$, and a point $u \in W^{1,m}(\Omega)$ such that $u_j \to u$, weakly in $W^{1,m}(\Omega)$. To show that $u_j \to u$ in $W^{1,m}(\Omega)$, we first assume $m \geq 2$ and set $h(u) = -u|u|^{m-2} + f(u)$. Since m - Laplacian is strongly monotone (see e.g., [3, Lemma 4.10], we have

$$
\begin{aligned}
a_2\varepsilon^m \|\nabla u_j - \nabla u\|_{L^m}^m & \\
&\leq \varepsilon^m \int_\Omega (|\nabla u_j|^{m-2}\nabla u_j - |\nabla u|^{m-2}\nabla u)(\nabla u_j - \nabla u)\, dx \\
&= \langle DJ_\varepsilon(u), u_j - u \rangle + \langle DJ_\varepsilon(u_j), u_j - u \rangle \\
&\quad + \int_\Omega (h(u_j) - h(u_j))(u_j - u)\, dx.
\end{aligned} \tag{2.1}
$$

Applying (f_3), Hölder's inequality and Sobolev imbedding theorem, we can show that the right side of (2.1) tends to 0 as $j \to \infty$. For the case $1 < m < 2$, let $H_j(x) = (|\nabla u_j|^{m-2}\nabla u_j - |\nabla u|^{m-2}\nabla u)(\nabla u_j - \nabla u)$. Due to $u_j \to u$ weakly and [3, Lemma 4.10], $\int_\Omega H_j\, dx \to 0$ and

$$(|\nabla u_j - \nabla u|^2) \leq H_j(x)(|\nabla u_j| + |\nabla u|)^{2-m}.$$

Hence we have, from Hölder's inequality,

$$\int_\Omega |\nabla u_j - \nabla u|^m\, dx \leq \left(\int_\Omega H_j\, dx\right)^{\frac{m}{2}} \left(\int_\Omega (|\nabla u_j| + |\nabla u|)^m\, dx\right)^{\frac{2-m}{2}},$$

which shows $\|\nabla u_j - \nabla u\| \to 0$. Since $u_j \to u$ in $L_m(\Omega)$, $u_j \to u$ in $W^{1,m}(\Omega)$. This shows Palais Smale condition. Finally by (f_3), there is $a_3 > 0$,

$$\int_\Omega F(u)\, dx \leq \theta \int_\Omega |u|^m\, dx + a_3 \int_\Omega |u|^p\, dx$$

and

$$
\begin{aligned}
J_\varepsilon(u) &= \frac{1}{m}\|u\|_{W^{1,m}}^m - \int_\Omega F(u)\, dx \\
&\geq (\frac{1}{m} - \theta)\|u\|_{W^{1,m}}^m - a_3\|u\|_{W^{1,m}}^p > 0
\end{aligned}
$$

for sufficiently small $\|u\|_{W^{1,m}}$. Hence by the mountain pass lemma, [8, Theorem 2.2], c_ε in (1.2) is a positive critical value. □

We now call the critical point u_ε corresponding to c_ε by the least energy solution. To show the reason, we need the following lemma

LEMMA 2.2. *Let $v \in W^{1,m}(\Omega)$ and $v \neq 0$ and define*

$$h_d(t) = t^{m-1} \int_\Omega d|\nabla v|^m + |v|^m \, dx - \int_\Omega v f(tv) \, dx \,, \tag{2.2}$$

where d is a positive constant, then there exists unique $\lambda > 0$ satisfying $h_d(\lambda) = 0$ such that $h(t) > 0$ for $0 < t < \lambda$ and $h_d(t) < 0$ for $t > \lambda$.

Proof. Clearly $h_d(t)$ is continuous and

$$\frac{h_d(t)}{t^{m-1}} = \int_\Omega d|\nabla v|^m + |v|^m \, dx - \int_\Omega |v|^m \frac{f(tv)}{|tv|^{m-1}} \, dx.$$

Noting that the last integral is strictly increasing in t by (f_2) and taking (f_3) into account, it is easy to check the assertion. □

THEOREM 2.3. *Let e be non trivial such that $J_\varepsilon(e) = 0$ and let c_ε be a critical value corresponding to e, which is determined by Mountain Pass lemma. Then c_ε is characterized by*

$$c_\varepsilon = \inf\{\sup_{t \geq 0} J_\varepsilon(tv) \,|\, v \in W^{1,m}(\Omega), \; v \neq 0 \;\; \text{and } v \geq 0\}. \tag{2.3}$$

Proof. Let u_ε be a critical point corresponding to c_ε. Then by Lemma 2.2, with $d = \varepsilon^m$ in (2.2),

$$c_\varepsilon = \sup_{t \geq 0} J_\varepsilon(tu_\varepsilon)$$

Hence

$$c_\varepsilon \geq \inf\{\sup_{t \geq 0} J_\varepsilon(tv) \,|\, v \in W^{1,m}(\Omega), \, v \neq 0 \text{ and } v \geq 0\}.$$

Now suppose that there exists nontrivial and nonnegative $\tilde{v}$ with $J_\varepsilon(\tilde{v}) = 0$ such that

$$c_\varepsilon > \sup_{t \geq 0} J_\varepsilon(\tilde{v}).$$

By Lemma 2.2, there exists $\tilde{t} > 0$ such that $\tilde{e} = \tilde{t}\tilde{v}$ satisfies $J_\varepsilon(\tilde{e}) = 0$. Let $V = \{ae + b\tilde{e} \mid a, b \in \mathbf{R}\}$ and $V^+ = \{ae + b\tilde{e} \mid a, b \in \mathbf{R}^+\}$. Let us choose a ball B_R with $R \geq \max\{\|e\|_{W^{1,m}(\Omega)}, \|\tilde{e}\|_{W^{1,m}(\Omega)}\}$ such that on $\partial B_R \cap V^+$, $J_\varepsilon \leq 0$. Let γ be a path consisting of line segments 0 to $R\tilde{e}/\|\tilde{e}\|$ and the arc $\partial B_R \cap V^+$ and the line segment $Re/\|e\|$ to e. Clearly $\gamma \in \Gamma$ and it is easy to see that $c_\varepsilon > \sup_{v \in \gamma} J_\varepsilon(v)$. But it is against the definition of c_ε. □

REMARK. Theorem 2.3 shows that c_ε is independent on the choice of e. Moreover, it is the least energy in the sense that if v is another solution then $J_\varepsilon(v) \geq c_\varepsilon\,(= J_\varepsilon(u_\varepsilon))$.

3. The asymptotic behavior of the least energy solution

We now define an (internal) energy functional by

$$E_\varepsilon(v) = \int_\Omega \varepsilon^m |\nabla v|^m + |v|^m \, dx.$$

Note that if u is a critical point of J_ε, u satisfies the following relation, which easily comes from $\langle DJ_\varepsilon(u), u\rangle = 0$.

$$E_\varepsilon(u) = \int_\Omega u f(u)\, dx.$$

THEOREM 3.1. *Let u_ε be a least energy solution to* (I_ε). *Then there exist positive constants b_1 and b_2 such that for all sufficiently small $\varepsilon > 0$,*

$$b_1\varepsilon^N < E_\varepsilon(u_\varepsilon) < b_2\varepsilon^N.$$

Proof. Let v be a critical point of J_ε. Set $\Omega_\varepsilon = \{\, x \mid \varepsilon x \in \Omega\}$ and define $w(x)$ by $v(x) = w(\varepsilon^{-1}x)$. Then for any q with $m \leq q < \frac{mN}{N-m}$, applying the Sobolev imbedding theorem,

$$\begin{aligned}
\left(\varepsilon^{-N}E_\varepsilon(v)\right)^{\frac{1}{m}} &= \left(\int_{\Omega_\varepsilon} |\nabla w|^m + w^m\, dx\right)^{\frac{1}{m}} \\
&\geq b_q \left(\int_{\Omega_\varepsilon} |w|^q\, dx\right)^{\frac{1}{q}} \\
&= b_q \left(\varepsilon^{-N}\int_\Omega |v|^q\, dx\right)^{\frac{1}{q}}
\end{aligned}$$

Hence

$$\varepsilon^{-N}\int_\Omega |v|^q\, dx \leq b_q^{-q}\left(\varepsilon^{-N}E_\varepsilon(v)\right)^{\frac{q}{m}} \tag{3.1}$$

On the other hand by (f_3), there exists $\alpha > 0$ such that

$$\varepsilon^{-N}E_\varepsilon(v) = \varepsilon^{-N}\int_\Omega vf(v)\,dx$$

$$\leq o(1)\varepsilon^{-N}\int_\Omega v^m\,dx + \alpha\varepsilon^{-N}\int_\Omega v^q\,dx \tag{3.2}$$

by (3.1) and (3.2) it is easily seen that

$$\left(\varepsilon^{-N}E_\varepsilon(v)\right)^{q/m-1} \geq b_q^q\,/\alpha(1-o(1)b_m^{-m}).$$

Since $\frac{q}{m}-1>0$ and b_m, b_q depend on only q, m, N and the cone property of Ω, (see e.g., [1, Lemma 5.14]), the left inequality holds. On the other hand, from (f_4),

$$J_\varepsilon(u_\varepsilon) \geq (\frac{1}{m}-\theta)\int_\Omega u_\varepsilon f(u_\varepsilon)\,dx$$

$$= (\frac{1}{m}-\theta)E_\varepsilon(u_\varepsilon).$$

Hence

$$E_\varepsilon(u_\varepsilon) \leq (\frac{1}{m}-\theta)^{-1}J_\varepsilon(u_\varepsilon). \tag{3.3}$$

Now we may assume $0\in\Omega$ without loss of generality Set $\Omega_\varepsilon = \{x : \varepsilon x\in\Omega\}$. Then for all ε sufficiently small, the unit ball $B_1\subset\Omega_\varepsilon$. Now choose a particular function ψ given by

$$\psi(x) = \begin{cases} 1-|x| & \text{for } |x|\leq 1 \\ 0 & \text{for } |x|>1 \end{cases}$$

Clearly by Lemma 2.2, there exists $\lambda\geq 0$ independent of ε such that

$$\int_{\Omega_\varepsilon} |\lambda\nabla\psi|^m + |\lambda\psi|^m\,dx = \int_{\Omega_\varepsilon} f(\lambda\psi)\lambda\psi\,dx.$$

Let $v_\varepsilon(x) = \lambda\psi(\varepsilon^{-1}x)$. Then $\mathrm{Supp}(v_\varepsilon)\subset\Omega$, $E_\varepsilon(v_\varepsilon) = \int_\Omega f(v_\varepsilon)v_\varepsilon\,dx$ and for $b = \int_{\mathbf{R}^N} |\lambda\nabla\psi|^m + |\lambda\psi|^m\,dx$,

$$E_\varepsilon(v_\varepsilon) = \varepsilon^N\left(\int_{\Omega_\varepsilon} |\lambda\nabla\psi|^m + |\lambda\psi|^m\,dx\right) \leq b\varepsilon^N. \tag{3.4}$$

On the other hand,

$$J_\varepsilon(u_\varepsilon) \le \sup_{t>0} J_\varepsilon(tv_\varepsilon) = J_\varepsilon(v_\varepsilon) \le \frac{1}{m} E_\varepsilon(v_\varepsilon). \tag{3.5}$$

Hence from (3.3), (3.4) and (3.5), we obtain the right inequality of (3.1), with $b_2 = \frac{1}{m}(\frac{1}{m} - \theta)^{-1} b$. □

PROPOSITION 3.2. *Let u_ε be a least energy solution corresponding to c_ε defined by (1.2). Then u_ε is positive.*

Proof. Suppose to the contrary that $|\{x : u_\varepsilon \le 0\}| > 0$. Then u_ε is changing sign solution i.e., $|\{x : u_\varepsilon > 0\}| > 0$ since $u_\varepsilon^+ \equiv 0$ yields absurdity that $0 < E_\varepsilon(u_\varepsilon) = \int_\Omega u_\varepsilon^+ f(u_\varepsilon^+)\, dx = 0$. Clearly for all $t > 0$,

$$\begin{aligned} J_\varepsilon(tu_\varepsilon^+) &< J_\varepsilon(tu_\varepsilon) \\ &\le \sup_{t>0} J_\varepsilon(tu_\varepsilon) \\ &= J_\varepsilon(u_\varepsilon) = c_\varepsilon. \end{aligned}$$

Hence by Theorem 2.3, there comes out a contradiction ;

$$c_\varepsilon \le \sup_{0 \le t} J_\varepsilon(tu_\varepsilon^+) < c_\varepsilon.$$

Hence u_ε is nonnegative. Now by the regularity result in [5, 9], u_ε is $C^{1,\alpha}$ and it is easy to see that $u_\varepsilon > 0$ using the extended version of Hopf boundary point lemma, which can be seen in [4, 10]. □

COROLLARY 3.3. *For all sufficiently small ε, each least energy solution is not constant*

Proof. If λ is a constant solution, λ satisfy

$$\lambda^{m-1} = f(\lambda).$$

By (f_2) and (f_3), such λ is unique. Then for all small $\varepsilon > 0$, we have

$$E_\varepsilon(u_\varepsilon) = \lambda|\Omega| > b_2 \varepsilon^N.$$

But it is against theorem 3.1. □

THEOREM 3.4. *Let u_ε be a least energy solution to (I_ε). Then there exist positive constants c_q, C_q and C_∞ independent on ϵ such that for $q \geq m$*

$$c_q\varepsilon^N \leq \int_\Omega |u_\varepsilon|^q \, dx \leq C_q\varepsilon^N \tag{3.6}$$

$$\|u\|_\infty \leq C_\infty. \tag{3.7}$$

Proof. We write u_ε by u for simplicity. Now we apply the idea of Brezis and Kato which can be seen in [2]. Recall that each solution u of (I_ε) satisfy

$$\int_\Omega \varepsilon^m |\nabla u|^{m-2}\nabla u \nabla v + \int_\Omega u^{m-1} v = \int_\Omega f(u) v \, dx$$

for all $v \in W^{1,m}(\Omega)$. Now choose a sequence $(\ell_j)_{j\geq 0}$ satisfying

$$\begin{cases} \ell_0 = 1 \\ (\ell_j - 1)m = (\frac{mN}{N-m})\ell_{j-1} - p \qquad j = 1, 2, \ldots \end{cases}$$

First take $v = u^{(\ell_1 - 1)m+1}$. Then we obtain

(3.8)
$$\int_\Omega u^{\ell_1 m} \, dx + \varepsilon^m \frac{(\ell_1 - 1)m + 1}{\ell_1^m} \int_\Omega |\nabla u^{\ell_1}|^m \, dx = \int_\Omega u^{(\ell_1 - 1)m+1} f(u) \, dx.$$

And by (f_3), there exists a positive constant $c = c(f)$ such that

$$\int_\Omega u^{(\ell_1 - 1)m+1} f(u) \, dx \leq o(1) \int_\Omega u^{\ell_1 m} \, dx + c \int_\Omega u^{j_0 m^*} \, dx. \tag{3.9}$$

By (3.8) and (3.9), we have

$$\int_\Omega \varepsilon^m |\nabla u^{\ell_1}|^m + u^{\ell_1 m} \, dx \leq c\ell_1^m \int_\Omega u^{m^*} \, dx.$$

This shows $u^{\ell_1} \in W^{1,m}$ by virtue of Sobolev imbedding theorem. Repeat this process (substituting $u^{(\ell_j - 1)m+1}$ for v) successively for the sequence $(\ell_j)_{j\geq 1}$ to obtain

$$\int_\Omega \varepsilon^m |\nabla u^{\ell_j}|^m + u^{\ell_j m} \, dx \leq c\ell_j^m \int_\Omega u^{m^* \ell_{j-1}} \, dx. \tag{3.10}$$

This shows $u^{\ell_j} \in W^{1,m}(\Omega)$ and hence $L_{m^*\ell_j}(\Omega)$ for all ℓ_j.

Now we can give the proof of (3.7). Apply Sobolev imbedding theorem to the left hand side of (3.10), then there exists $\tilde{c}$ such that

$$(3.11)\qquad \varepsilon^m \left(\int_\Omega u^{(\frac{mN}{N-m})\ell_j}\, dx\right)^{\frac{N-m}{N}} \le \tilde{c}\int_\Omega \varepsilon^m|\nabla u^{\ell_j}|^m + u^{\ell_j m}\, dx.$$

Let $I_j = \int_\Omega u^{\ell_j m^*}\, dx$ and $\alpha = \frac{N}{n-m}$. Then (3.10) and (3.11) yields

$$(3.12)\qquad I_j \le (c^*\varepsilon^{-m}\ell_j^m I_{j-1})^\alpha \qquad j = 1, 2, \cdots .$$

Now by Theorem 3.1 and Sobolev imbedding theorem, we have

$$(3.13)\qquad I_0 = \int_\Omega u^{m^*}\, dx \le c_0\varepsilon^N$$

From (3.12) and (3.13),

$$\begin{aligned} I_1 &\le (c^*c_0\varepsilon^{-m}\ell_1 I_0)^\alpha \\ &\le c_1\ell_1^{m\alpha}\varepsilon^N, \qquad c_1 = (c^*c_0)^\alpha. \end{aligned}$$

Inductive process shows

$$(3.14)\qquad I_j \le \left(c_j\ell_j^{m\alpha}\,\ell_{j-1}^{m\alpha^2}\cdots\ell_1^{m\alpha^j}\right)\varepsilon^N, \quad j = 0, 1, 2, \cdots ,$$

where $c_j = c_0^{\alpha^j}c^{*\sum_{k=1}^{k=j}\alpha^k}$. Clearly there exist positive constants a, A with $a < 1 < A$ such that

$$(3.15)\qquad a\alpha^j \le \ell_j \le A\alpha^j.$$

From (3.14) and (3.15), simple calculation shows

$$\|u\|_{L_{\ell_j m^*}} = I_j^{\frac{1}{\ell_j m^*}} \le C\varepsilon^{\frac{N}{\ell_j m^*}} A^{\frac{1}{a}\frac{N-m}{m}} \left(\frac{N}{N-m}\right)^{\frac{N(N-m)^2}{am}}$$

where $C = \left((\max(1, c^*))^{\frac{N}{m}}c_0\right)^{\frac{1}{am^*}}$. By letting $j \to \infty$, we can find (3.7) with $C_\infty = CA^{\frac{1}{a}\frac{N-m}{m}}\left(\frac{N}{N-m}\right)^{\frac{N(N-m)^2}{am}}$. Now it remains to show

(3.6). To see the right inequality of (3.6), note that $\int_\Omega u^m dx \le C\varepsilon^N$ by (3.1) and from (3.14),

$$\int_\Omega u^{\ell_j m^*}\, dx \le C_j \varepsilon^N \qquad j = 0, 1, 2, \ldots,$$

where $C_0 = c_0$ and $C_j = c_j \ell_j^{m\alpha} \ell_{j-1}^{m\alpha^2} \cdots \ell_1^{m\alpha^j}$. Then standard L_q interpolation yields the result. Finally for the left inequality, apply (f_3) and Theorem 3.4 to the equation:

$$E_\varepsilon(u) = \int_\Omega u f(u)\, dx.$$

Then we easily obtain the inequality for $q = p$. Now for $m \le q \le p$, we have from (3.7)

$$c_p \varepsilon^N \le \int_\Omega u^p\, dx = \int_\Omega u^{p-q} u^q\, dx \le C_\infty^{p-q} \int_\Omega u^q\, dx.$$

Finally for $q > p$ write $p = (1-\lambda)m + \lambda q$. Then

$$\begin{aligned} c_p \varepsilon^N &\le \int_\Omega u^p\, dx = \int_\Omega u^{(1-\lambda)m+\lambda q}\, dx \\ &\le \Big(\int_\Omega u^m\, dx\Big)^{1-\lambda} \Big(\int_\Omega u^q\, dx\Big)^\lambda \\ &\le c_m^{1-\lambda} \varepsilon^{N(1-\lambda)} \Big(\int_\Omega u^q\, dx\Big)^\lambda. \end{aligned}$$

Therefore we obtain

$$\int_\Omega u^q\, dx \ge c_m^{1-1/\lambda} c_q^{1/\lambda} \varepsilon^N.$$

This completes the proof. □

ACKNOWLEDGEMENT. The first author is partly supported by KOSEF, GARC, BSRI and YSU Grant '95. The second author is partly supported by KRF.

References

1. R. A. Adams, *Sobolev spaces*, Academic Press, 1975.
2. H. Brezis, T. Kato, *Remarks on the Schrödinger operator with singular complex potential*, J. Math. Pures et Apple. **58** (1979), 137-151.
3. J. I. Díaz, *Nonlinear partial differential equation and free boundaries* **I**, Pitman Advanced Publishing Program, 1985.
4. M. Guedda, L. Veron, *Quasilinear equations involving critical Sobolev exponents*, Nonlinear Analysis, Theory and Methods.
5. Gray M Liberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Analysis, Theory and Methods and Appl. **12-11** (1988), 1203-1219.
6. C. H. Lin, W. M-Ni, I. Takagi, *Large amplitude stationary solutions to a chemotaxis system*, J. Diff. Eq. **72** (1988), 1-27.
7. W. M-Ni, I. Takagi, *On the shape of least energy solutions to a semilinear Neumann problem*, Comm. Pure Appl. Math. **XLIV** (1991), 819-851.
8. Paul H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, AMS **65**.
9. P. Torksdorf, *Regularity for more general class of quasilinear elliptic equation*, J. Diff. Eq. **51** (1984), 126-150.
10. J. L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), 191-202.

Department of Mathematics
Yonsei University
Seoul 120-749, Korea