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THE INVARIANCE PRINCIPLE FOR ρ–MIXING RANDOM FIELDS

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1. Introduction

Ibragimov(1975) showed the central limit theorem and the invariance principle for $\rho-$mixing random variables satisfying $\sigma^2(n) = nh(n) \to \infty$ and $E|\xi_0|^{2+\delta} < \infty$ for some $\delta > 0$ where $\sigma^2(n)$ denotes the variance of the partial sum S_n and $h(n)$ is a slowly varying function.

In this paper we extend the concept of $\rho-$mixing to random fields and obtain an invariance principle for such random fields. This theorem may be regarded as a generalization of Theorem 3.1 of Ibragimov(1975) to multivariate time.

Let Z^d denote the set of all $d-$tuples of integers($d \geq 1$), and let $\{\xi_{n_1,n_2,\cdots,n_d} : (n_1, n_2, \cdots, n_d) \in Z^d\}$ be a random field, i.e., a collection of random variables indexed by time set Z^d. For each $j(1 \leq j \leq d)$ and $r \geq 0$, let $\mathcal{A}^+(j;r)$ be the $\sigma-$field generated by $\{\xi_{n_1,n_2,\cdots,n_d} : n_j \geq r,$ other $n_i's$ unrestricted$\}$ and let $\mathcal{A}^-(j;r)$ be the $\sigma-$field generated by $\{\xi_{n_1,n_2,\cdots,n_d} : n_j \leq r,$ other $n_i's$ unrestricted$\}$ and we shall write $L(\mathcal{A}^+(j;r))[L(\mathcal{A}^-(j;r))]$ for the collection of all $\mathcal{A}^+(j;r)[\mathcal{A}^-(j;r)]$ measurable random variables with finite variance. For $r \geq 1$, we write

$$\rho(j;r) = \sup \frac{E[(x-Ex)(y-Ey)]}{E^{\frac{1}{2}}(x-Ex)E^{\frac{1}{2}}(y-Ey)}, \tag{1.1}$$

where the supremum is taken over all random variables $x \in L(\mathcal{A}^+(j;r))$, $y \in L(\mathcal{A}^-(j;r))$,and

$$\rho(r) = \max_{i \leq j \leq d} \rho(j;r). \tag{1.2}$$

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Clearly $\{\rho(r)\}$ is a decreasing sequence of real numbers. If $\rho(r) \to 0$ we say that the random field is ρ-mixing. This is a natural extension to multivariate time parameter of the well known concept of ρ-mixing for sequences of random variables.

Let T^d be the d-fold product of the closed unit interval $[0,1]$, let D_d be the Skorokhod function space on T^d; and let $\{W(t_1, t_2, \cdots, t_d) : (t_1, t_2, \cdots, t_d) \in T^d\}$ be the d-parameter Wiener process. Assume $E(\xi_{0,0,\cdots,0}) = 0, E(\xi^2_{0,0,\cdots,0}) < \infty$. For $n_1 \geq 1, n_2 \geq 1, \cdots, n_d \geq 1$, define the partial sum,

$$S_{n_1,n_2,\cdots,n_d} = \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \cdots \sum_{j_d=1}^{n_d} \xi_{j_1,j_2,\cdots,j_d}. \tag{1.3}$$

If some $n_i = 0$ and other n_i are greater than or equal to 0, it is convenient to set $S_{n_1,n_2,\cdots,n_d} = 0$. Let $\sigma^2(n_1, n_2, \cdots, n_d)$ denote the variance of $S_{n_1,n_2,\cdots,n_d}$. Construct a sequence $\{W_n(t_1, t_2, \cdots, t_d) : n \geq 1, (t_1, t_2, \cdots, t_d) \in T^d\}$ of stochastic processes in D_d by

$$W_n(t_1, t_2, \cdots, t_d) = \sigma(n, n, \cdots, n)^{-1} S_{[nt_1],[nt_2],\cdots,[nt_d]}, \tag{1.4}$$

$(t_1, t_2, \cdots, t_d) \in T^d$, where $[\cdot]$ is the usual greatest integer function.

In the case of univariate time parameter, i.e., $d = 1$, the invariance principle has been proved under two different sets of hypotheses. In one case no assumption is made about existence of a moment of ξ_0 of order greater than two; however a condition is imposed on the rate at which the mixing coefficient $\rho(r)$ goes to zero. Ibragimov(1975), however, had a theorem([8], Theorem 2.1), in which convergence of normalized partial sums of ρ-mixing sequences to normal distribution is proved without assuming the condition imposed on the rate of ρ-mixing coefficient. What is assumed instead is

$$E(|\xi_0|^{2+\delta}) < \infty \quad \text{for some} \quad \delta > 0, \quad \sigma^2(n) = Var(S_n) \to \infty. \tag{1.5}$$

This theorem was casted in the form of the invarance principle by Ibragimov(1975).

THEOREM 0. (Ibragimov, 1975) *Let $\{\xi_n : n \in N\}$ be a ρ-mixing processes with $E\xi_n = 0$. If (1.5) is fulfilled then $\{\xi_n\}$ satisfies the invariance principle.*

The object of this note is to extend Theorem 0 to random fields, that is, to prove an invariance principle for general ρ-mixing random fields.

In Section 2 we state some assumptions and main theorem. The proofs of main theorem as well as lemmas are included in Section 3.

2. Preliminaries and main theorem

The main difficulty seems to be in the proper generalization of the one-dimensional condition that $\sigma^2(n) = Var(S_n) \to \infty$ as $n \to \infty$. A natural generalization would be to require that $\sigma^2(n_1, n_2, \cdots, n_d) = Var(S_{n_1,n_2,\cdots,n_d}) \to \infty$ as $(n_1, n_2, \cdots, n_d) \to \infty$, where $(n_1, n_2, \cdots, n_d) \to \infty$ means, as usual, $\min_{1\le i\le d} n_i \to \infty$. That this is not enough is shown, by the following example. Let $\{\eta_{m,n} : -\infty < m < \infty, -\infty < n < \infty\}$ be a collection of independent standard normal variables. Define $\xi_{m,n} = \eta_{m+1,n} - \eta_{m,n}$. Then $\{\xi_{m,n}\}$ is clearly a ρ-mixing, two-parameter, random field with $\xi_{m,n}$ having moments of all orders. Also $\sigma^2(m,n) = Var(S_{m,n}) = 2n$ which goes to infinity as $(m,n) \to \infty$. However in this case the limiting process of the sequence $\{W_n\}$ is not the two-parameter Wiener-process, but the degenerate process $\{\zeta(t_1,t_2) : 0 \le t_1, t_2 \le 1\}$ where $\zeta(t_1,t_2) = W(t_2)$, W being the standard one-parameter Brownian motion on $[0,1]$.

Thus what seems to be required is a condition which ensures that the variance structure of $S_{n_1,n_2,\cdots,n_d}$ is sufficiently homogeneous and the variances tend to infinity in all directions. The following condition accomplishes this. We assume that there exists a positive integer $\overline{N}$ such that for each $i, 1 \le i \le d$, the following is true:

$$(2.1) \qquad \lim_{n_i\to\infty} \frac{\sigma^2(n_1, n_2, \cdots, n_{i-1}, n_i, n_{i+1}, \cdots, n_d)}{\sigma^2(n_1, n_2, \cdots, n_{i-1}, 1, n_{i+1}, \cdots, n_d)} = \infty,$$

and that this limit is uniform in all values of $n_1, n_2, \cdots, n_{i-1}, n_{i+1}, \cdots, n_d$ greater than $\overline{N}$. Thus, e.g., if $d = 2$, we require that

$$(2.2) \qquad \lim_{n_1\to\infty} \frac{\sigma^2(n_1,n_2)}{\sigma^2(1,n_2)} = \infty \quad \text{for each} \quad n_2 \ge 1,$$

$$(2.3) \qquad \lim_{n_2\to\infty} \frac{\sigma^2(n_1,n_2)}{\sigma^2(n_1,1)} = \infty \quad \text{for each} \quad n_1 \ge 1,$$

and that the limit in (2.2) is uniform in $n_2 \geq \overline{N}$ and the limit in (2.3) is uniform in $n_1 \geq \overline{N}$. To put condition (2.1) in perspective, note that it is implied by:

$$\lim_{(n_1,n_2,\cdots,n_d)\to\infty} \frac{\sigma^2(n_1,n_2,\cdots,n_d)}{n_1 n_2 \cdots n_d} = \sigma^2 > 0. \tag{2.4}$$

To formulate an analogue of the Ibragimov theorem([7], Theorem3.1), we assume that $\{\xi_{n_1,n_2,\cdots,n_d} : (n_1,n_2,\cdots,n_d) \in Z^d\}$ is a ρ-mixing random field with $E(\xi_{0,0,\cdots,0}) = 0$ which satisfies (2.1) and the following condition,

$$E|\xi_{0,0,\cdots,0}|^{2+\delta} < \infty \quad \text{for some} \quad \delta > 0. \tag{2.5}$$

THEOREM 2.1. *Let $\{\xi_{n_1,n_2,\cdots,n_d} : (n_1,n_2,\cdots,n_d) \in Z^d\}$ be a ρ-mixing random field. If (2.1) and (2.5) are fulfilled then the sequence of stochastic processes $\{W_n\}$ satisfies the invariance principle.*

COROLLARY 2.2. *Let $\{\xi_{n_1,n_2,\cdots,n_d} : (n_1,n_2,\cdots,n_d) \in Z^d\}$ be a ρ-mixing random field with $E\xi_{0,0,\cdots,0} = 0$. If (2.4) and (2.5) are fulfilled then $\{W_n\}$ satisfies the invariance principle.*

As is shown in [4], since (2.4) itself is implied by

$$\begin{aligned} &\sum_{j_1=-\infty}^{\infty}\sum_{j_2=-\infty}^{\infty}\cdots\sum_{j_d=-\infty}^{\infty} |E(\xi_{0,0,\cdots,0}\xi_{j_1,j_2,\cdots,j_d}) < \infty \quad \text{and} \\ &\sum_{j_1=-\infty}^{\infty}\sum_{j_2=-\infty}^{\infty}\cdots\sum_{j_d=-\infty}^{\infty} E(\xi_{0,0,\cdots,0}\xi_{j_1,j_2,\cdots,j_d}) > 0, \end{aligned} \tag{2.6}$$

we have the following corollary :

COROLLARY 2.3. *Let $\{\xi_{j_1,j_2,\cdots,j_d} : (j_1,j_2,\cdots,j_d) \in Z^d\}$ be a ρ-mixing random field. If (2.5) and (2.6) are fulfilled then $\{\xi_{j_1,j_2,\cdots,j_d}\}$ satisfies the invariance principle.*

3. Proof of Theorem 2.1

To alleviate the notational burden we write out details for $d = 2$. For bigger d the proof is similar but more tedious. We assume conditions of Theorem 2.1 in all the following lemmas and write $h(n_1,n_2) = (n_1n_2)^{-1}\sigma^2(n_1,n_2)$.

LEMMA 3.1. *If $\sigma^2(n_1, n_2) \to \infty$ then (a), (b) and (c) below hold.*

(a) $\lim_{n_1\to\infty}[\frac{\sigma^2(c_1 n_1, n_2)}{\sigma^2(n_1, n_2)}] = c_1$, *uniformly in* $n_2 \geq \overline{N}$.

(b) $\lim_{n_2\to\infty}[\frac{\sigma^2(n_1, c_2 n_2)}{\sigma^2(n_1, n_2)}] = c_2$, *uniformly in* $n_1 \geq \overline{N}$.

(c) $\lim_{(n_1,n_2)\to\infty}[\frac{\sigma^2(c_1 n_1, c_2 n_2)}{\sigma^2(n_1, n_2)}] = c_1 c_2$,

where c_1, c_2 are positive integers.

Proof. The proof is similar to that of Lemma 1 of Deo[5].

LEMMA 3.2. *Given $\delta > 0$, we can find $N = N(\delta)$ such that*

$$\sup_{\substack{n_1\geq 1\\ n_2\geq 1}} \max_{\substack{0\leq c_1\leq 1\\ 0\leq c_2\leq 1}} c_1^\delta c_2^\delta \frac{h(c_1 n_1, c_2 n_2)}{h(n_1, n_2)} \leq \overline{M} < \infty,$$

where $\overline{M}$ is a finite number.

Proof. It can be easily proved by the similar way to the proof of Lemma 6 in [6].

LEMMA 3.3. *Under conditions of Theorem 2.1, if $\delta < 1$, then there exists $A > 0$ such that*

$$E|S_{n_1,n_2}|^{2+\delta} \leq A(\sigma(n_1, n_2))^{2+\delta} \quad \text{for all} \quad n_1, n_2. \tag{3.1}$$

Proof. Apply the arguments in the proof of Lemma 2.1 in [8] to the sequence $\{S_{n_1,n_2} - S_{n_1-1,n_2} : 1 \leq n_1 < \infty\}$ and note that the resulting constant A is independent of n_2 for all n_2. The details are straightforward and therefore omitted. The remaining finite number of n_2's can be handled by increasing A if necessary and applying the univariate-time Lemma 2.1 in [8].

LEMMA 3.4. *Under conditions of Theorem 2.1 there exist $B > 0, \gamma > 1$ and a positive integer L such that*

(3.2)

$$E|W_n(t_1, t_2)|^{2+\delta} \leq B(t_1 t_2)^\gamma \quad \text{for all} \quad 0 \leq t_1, t_2 \leq 1 \quad \text{and all} \quad n \geq L.$$

Proof. Combine the preceding lemmas with arguments on page 693 of Davydov [2]. Davydov uses Karamata's representation of slowly varying function. In our case this is not available, but use of Lemma 3.2 makes it unnecessary. According to the preceding lemmas we obtain the following inequality,

$$\begin{aligned} E|W_n(t_1,t_2)|^{2+\delta} &\leq A(\frac{\sigma^2([nt_1],[nt_2])}{\sigma^2(n,n)})^{\frac{2+\delta}{2}} \\ &= A(\frac{[nt_1][nt_2]h([nt_1],[nt_2])}{n^2h(n,n)})^{\frac{2+\delta}{2}} \\ &\leq B(t_1t_2)^{\gamma}. \end{aligned}$$

Proof of Theorem 2.1. The proof of this theorem can now be completed by applying Lemma 3 of Deo[5] to the sequence $\{W_n\}$. Condition (iv) of this lemma is satisfied by $\{W_n\}$ because of Lemma 3.4 here and the equation (1) and Theorem 1 of Bickel and Wichura[1]. That the conditions (ii) and (iii) of Lemma 3 of Deo[5] are satisfied by our sequence $\{W_n\}$ here is a straightforward verification and the condition (ii) of Lemma 3 of Deo[5] is also satisfied by (3.2). This completes the proof of Theorem 2.1.

4. Applications

If X is a random measure and B is a Borel subset of R^d then $X(B)$ denotes the mass that the random measure gives to B. All random measures will be assumed to be stationary. Let $I_{\underline{j}}$ be the unit interval $(\underline{j}-\underline{1},\underline{j}]$, where $\underline{j}-\underline{1}=(j_1-1,j_2-1,\cdots,j_d-1)$ and $\underline{j}=(j_1,j_2,\cdots,j_d),\underline{j}\in Z^d$. If $\{X(I_{\underline{j}}):\underline{j}\in Z^d\}$ satisfies (1.1) , (1.2) and $\rho(r)\to 0$ we say that random measure X is $\rho-$mixing. Let $X_K(\underline{t})=X_K(t_1,\cdots,t_d)$ be defined by

(4.1)
$$X_K(\underline{t})=\frac{X((0,Kt_1]\times\cdots\times(0,Kt_d])-EX((0,Kt_1]\times\cdots\times(0,Kt_d])}{s^2K^{\frac{d}{2}}}$$

for $\underline{t}\in T^d$, where

$$s^2=\lim_{(n_1,n_2,\cdots,n_d)\to\infty}\frac{\sum_{j_1=1}^{n_1}\cdots\sum_{j_d=1}^{n_d}[X(I_{\underline{j}})-EX(I_{\underline{j}})]^2}{n_1n_2\cdots n_d}, \quad (4.2)$$

THEOREM 4.1. *Let $\{X_K\}$ be a sequence of random measures. If $\{X_K\}$ fulfills* (4.2) *and the following conditions* (4.3) *and* (4.4) *then the random measure X satisfies the invariance principle.*

(4.3) $EX(B) = 0, EX^2(B) < \infty$ *for all Borel subsets* $B \in R^d$,

(4.4) $E|X(I_{\underline{j}}) - EX(I_{\underline{j}})|^{2+\delta} < \infty$ *for some* $\delta > 0$.

Proof. It is sufficient to show that $X_K(\underline{t})$ converges, weakly to the d-dimensional Wiener measure. By the definition (4.1)

$$X_K(\underline{t}) = \frac{\sum_{j_1=1}^{[Kt_1]} \cdots \sum_{j_d=1}^{[Kt_d]} [X(I_{\underline{j}}) - EX(I_{\underline{j}})]}{K^{\frac{d}{2}} s^2} + \frac{X(([K\underline{t}], K\underline{t}]) - EX(([K\underline{t}], K\underline{t}]}{K^{\frac{d}{2}} s^2}.$$

According to Theorem 2.1 the first term in the right hand side converges to the d-dimensional Wiener measure and the second term converges in probability to zero as K. Thus the proof is complete by Theorem 4.1 of Billingsley [2].

Finally, we apply Theorem 4.1 to Poisson center random measures. These are constructed as follow : let U be a stationary Poisson point random field with parameter ρ. Let $V = \{V_{\underline{x}} | \underline{x} \in R^d\}$ be a collection of i.i.d. random measures with $E[V_{\underline{x}}(R^d)] = \xi < \infty$.
Then we say that X is a cluster process with centers U and members V if

$$X(B) = \sum_{\underline{x}: U(\underline{x}) > 0} V_{\underline{x}}(B - \underline{x})$$

for each bounded Borel set B. We denote X by $[U, V]$. (see [10]) It is natural to hope that moment conditions on V will imply moment conditions on X regardless of "shape" of V in R^d. This is made precise in the following theorem :

THEOREM 4.2. *Let $X = [U, V]$ as above. If X is ρ-mixing and* $E[V_{\underline{x}}(R^d)]^{2+\delta}]$
$< \infty$ *then X satisfies the invariance principle.*

Proof. According to Theorem 3.1 of Burton and Kim [3]. For a rectangular box B in R^d and $0 \le \delta \le 2$ there exists a constant K

depending only on δ and $|B|$ so that

$$E|X(B)|^{2+\delta} \leq KE\left[(V_{\underline{x}}(R^d))^{2+\delta}\right] \quad \text{and}$$
$$E|X(B) - EX(B)|^{2+\delta} \leq K|B|^{\frac{\delta}{2}}.$$

Thus by Theorem 4.1 the proof is complete.

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