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COINCIDENCES OF COMPOSITES OF U.S.C. MAPS ON H–SPACES AND APPLICATIONS

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1. Introduction

Applications of the classical Knaster - Kuratowski - Mazurkiewicz (si- mply, KKM) theorem and the fixed point theory of multifunctions defined on convex subsets of topological vector spaces have been greatly improved by adopting the concept of convex spaces due to Lassonde [L1]. In this direction, the first author [P5] found that certain coincidence theorems on a large class of composites of upper semicontinuous multifunctions imply many fundamental results in the KKM theory.

On the other hand, the concept of convex spaces was extended by Horvath [H1-5] to spaces having certain families of contractible subsets or H-spaces. In this direction, a number of authors extended important results on convex spaces to those on H-spaces. See Bardaro and Ceppitelli [BC1-3], Ding *et al.* [Di, DKT1-2, DTr], Park [P1-3], Tarafdar [T], and H. Kim [K].

In the present paper, we extend the main coincidence theorem in [P5] to H-spaces and apply it to obtain a far-reaching generalization of the KKM theorem, fixed point or coincidence theorems for H-spaces, and other results. We also obtain open-valued versions of a KKM theorem and a coincidence theorem for H-spaces. Many of the main results in the above-mentioned papers are extended and unified. Especially, the main theorems of [Di] are improved and corrected.

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2. Preliminaries

For the terminology and notations, we follow mainly [P1-5].

A *multifunction* (simply, *map*) $F : X \multimap Y$ is a function from a set X into the power set of a set Y. Let $F(A) = \bigcup\{Fx : x \in A\}$ for $A \subset X$ and $F^{-}y = \{x \in X : y \in Fx\}$ for $y \in Y$. As usual, the set $\mathrm{Gr}(F) = \{(x, y) : y \in Fx\}$ is called the *graph* of F.

For topological spaces X and Y, a map $F : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if for each closed set $B \subset Y$, $F^{-}(B) = \{x \in X : Fx \cap B \neq \emptyset\}$ is closed in X; and *compact* if $F(X)$ is contained in a compact subset of Y.

A subset C of a topological space X is said to be *compactly closed* [resp. *open*] in X if for every compact set $K \subset X$, the set $C \cap K$ is closed [resp. open] in K. A topological space X is said to be *contractible* if the identity map 1_X of X is homotopic to a constant map; and *acyclic* if all of its reduced Čech homology groups over rationals vanish.

Int and $\overline{}$ denote the interior and closure, resp.

For a set D, let $\langle D \rangle$ denote the set of all nonempty finite subsets of D.

Let X be a set (in a vector space) and D a nonempty subset of X. Then (X, D) is called a *convex space* [P5] if for each $N \in \langle D \rangle$, its convex hull $\mathrm{co}\, N$ is contained in X and X has a topology that induces the Euclidean topology on such convex hulls. Such convex hull will be called a *polytope*. A subset A of X is said to be *D-convex* if, for any $N \in \langle D \rangle$, $N \subset A$ implies $\mathrm{co}\, N \subset A$. If $X = D$, then $X = (X, X)$ becomes a convex space in the sense of Lassonde [L1].

A triple $(X, D; \Gamma)$ is called an *H-space* [P1,2] if X is a topological space, D a nonempty subset of X, and $\Gamma = \{\Gamma_A\}$ a family of contractible subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$. (The triple is called a *c-space* in [H5] whenever $X = D$.) If $X = D$, we denote $(X; \Gamma)$ instead of $(X, X; \Gamma)$. For an H-space $(X; \Gamma)$ and any nonempty subset Y of X, we have an H-space $(X, Y; \Gamma)$.

Any convex space (X, D) is an H-space $(X, D; \Gamma)$ by putting $\Gamma_A = \mathrm{co}\, A$ for $A \in \langle D \rangle$.

For an $(X, D; \Gamma)$, a subset C of X is said to be *H-convex* if for each $A \in \langle D \rangle$, $A \subset C$ implies $\Gamma_A \subset C$. A subset L of X is called an *H-subspace* of $(X, D; \Gamma)$ if $L \cap D \neq \emptyset$ and for every $A \in \langle L \cap D \rangle$,

$\Gamma_A \cap L$ is contractible. This is equivalent to saying that the triple $(L, L \cap D; \{\Gamma_A \cap L\})$ is an H-space.

Given a class $\mathbb{X}$ of maps, $\mathbb{X}(X, Y)$ denotes the set of all maps $F : X \multimap Y$ belonging to $\mathbb{X}$, and $\mathbb{X}_c$ the set of all finite composites of maps in $\mathbb{X}$.

For topological spaces X and Y, we define

$f \in \mathbb{C}(X, Y) \iff f$ is a (single-valued) continuous function.

$T \in \mathbb{K}(X, Y) \iff T$ is a *Kakutani map*; that is, Y is a convex space and T is u.s.c. with nonempty compact convex values.

$T \in \mathbb{V}(X, Y) \iff T$ is an *acyclic map*; that is, T is u.s.c. with compact acyclic values.

A class $\mathfrak{A}$ of maps is one satisfying:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and
(iii) for any polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point.

Examples of $\mathfrak{A}$ are $\mathbb{C}, \mathbb{K}, \mathbb{V}$, the Aronszajn maps $\mathbb{M}$ (with R_δ values) [Gr], the O'Neill maps $\mathbb{N}$ (with values consisting of one or m acyclic components, where m is fixed) [Gr], the approachable maps in topological vector spaces [BD1,2], the admissible maps of Górniewicz [Go], the permissible maps of Dzedzej [Dz], and others.

Further, we define the following:

$T \in \mathfrak{A}_c^\sigma(X, Y) \iff$ for any σ-compact subset K of X, there is a $\tilde{T} \in \mathfrak{A}_c(K, Y)$ such that $\tilde{T}x \subset Tx$ for each $x \in K$.

$T \in \mathfrak{A}_c^\kappa(X, Y) \iff$ for any compact subset K of X, there is a $\tilde{T} \in \mathfrak{A}_c(K, Y)$ as above.

The class $\mathbb{K}_c^+$ due to Lassonde [L3] and $\mathbb{V}_c^+$ due to Park, Singh, and Watson [PSW] belong to $\mathfrak{A}_c^\sigma$. Note that $\mathfrak{A} \subset \mathfrak{A}_c \subset \mathfrak{A}_c^\sigma \subset \mathfrak{A}_c^\kappa$. For details, see Park [P5] or [PK].

3. Coincidence theorems for H-spaces

In this section, we give a basic coincidence theorem.

Let Δ_n denote an n-simplex. We need the following:

LEMMA. *Let $(X, D; \Gamma)$ be an H-space where $D = \{x_0, x_1, \ldots, x_n\} \in \langle X \rangle$. Then there exists an $f \in \mathbb{C}(\Delta_n, X)$ such that $f(\Delta_J) \subset \Gamma_J$ for each $J \in \langle D \rangle$, where Δ_J is the face of Δ_n corresponding to J.*

Lemma is given implicitly by Horvath [H2, Théorème 1], [H3, Theorem 1] and explicitly by Ding and Tan [DT, Lemma 1] and Horvath [H4, Théorème 1], [H5, Theorem 1.1].

Our main result is the following general coincidence theorem related to the class $\mathfrak{A}_c^\kappa$:

THEOREM 1. *Let $(X, D; \Gamma)$ be an H-space, Y a Hausdorff space, $F \in \mathfrak{A}_c^\kappa(X, Y)$, and K a nonempty compact subset of Y. Let $S : D \multimap Y$ and $T : X \multimap Y$ satisfy the following:*

(1.1) *for each $x \in D$, Sx is compactly open in Y;*
(1.2) *for each $y \in F(X)$, $M \in \langle S^- y\rangle$ implies $\Gamma_M \subset T^- y$;*
(1.3) *$\overline{F(X)} \cap K \subset S(D)$; and*
(1.4) *either*
 (i) *$Y \backslash K \subset S(M)$ for some $M \in \langle D\rangle$; or*
 (ii) *for each $N \in \langle D\rangle$, there exists a compact H-subspace L_N of X containing N such that $F(L_N)\backslash K \subset S(L_N \cap D)$.*

Then T and F has a coincidence point $\bar{x} \in X$; that is, $T\bar{x} \cap F\bar{x} \neq \emptyset$.

Proof. Since $\overline{F(X)} \cap K$ is compact and covered by compactly open sets Sx by (1.1) and (1.3), there exists an $N \in \langle D\rangle$ such that $\overline{F(X)} \cap K \subset S(N)$.

Case (i). Since $Y\backslash K \subset S(M)$ for some $M \in \langle D\rangle$ by (i), we have $\overline{F(X)} \subset S(A)$ where $A = M \cup N = \{x_0, x_1, \cdots, x_n\} \in \langle D\rangle$. Then, by Lemma, there exists an $f \in \mathbb{C}(\Delta_n, X)$ such that $f(\Delta_n) \subset \Gamma_A$ and $f(\Delta_J) \subset \Gamma_J$ for each $J \in \langle A\rangle$, where $\Delta_n = \mathrm{co}\{e_0, e_1, \cdots, e_n\}$ and Δ_J is the face of Δ_n corresponding to J. Since $f(\Delta_n)$ is compact in X and $F \in \mathfrak{A}_c^\kappa(X, Y)$, there exists a $\tilde{F} \in \mathfrak{A}_c(f(\Delta_n), Y)$ such that $\tilde{F}x \subset Fx$ for each $x \in f(\Delta_n)$. Then $\tilde{F}f(\Delta_n)$ is compact in $\overline{F(X)}$, since it is the image of the compact set $f(\Delta_n)$ under the compact valued u.s.c. map $\tilde{F}$. Let $\{\lambda_i\}_{i=0}^n$ be the partition of unity subordinated to the cover $\{Sx_i \cap \tilde{F}f(\Delta_n)\}_{i=0}^n$ of $\tilde{F}f(\Delta_n)$.

Define a continuous map $p : \tilde{F}f(\Delta_n) \to \Delta_n$ by

$$py = \sum_{i=0}^{n} \lambda_i(y) e_i = \sum_{i \in N_y} \lambda_i(y) e_i \quad \text{for} \quad y \in \tilde{F}f(\Delta_n),$$

where $i \in N_y \iff \lambda_i(y) \neq 0 \implies y \in Sx_i \iff x_i \in S^- y$. By (1.2), we have $fp(y) \in f(\Delta_{N_y}) \subset \Gamma_{N_y} \subset T^{-}y$ for each $y \in \tilde{F}f(\Delta_n)$; that is, $y \in (Tfp)y$.

Since $p\tilde{F}f \in \mathfrak{A}_c(\Delta_n, \Delta_n)$, $p\tilde{F}f$ has a fixed point $z \in \Delta_n$; that is, $z \in (p\tilde{F}f)z$. Put $\bar{x} = fz$. Since $p^{-}z \cap (\tilde{F}f)z = p^{-}z \cap \tilde{F}\bar{x} \neq \emptyset$, for any $y \in p^{-}z \cap \tilde{F}\bar{x}$, we have $y \in \tilde{F}f(\Delta_n)$, $(fp)y = fz = \bar{x}$, and $y \in (Tfp)y = T\bar{x}$. Therefore, $p^{-}z \cap \tilde{F}\bar{x} \subset T\bar{x}$ and hence $T\bar{x} \cap \tilde{F}\bar{x} \subset T\bar{x} \cap F\bar{x} \neq \emptyset$.

Case (ii). For an $N \in \langle D \rangle$ such that $\overline{F(X)} \cap K \subset S(N)$, consider the set L_N in (1.4).

We claim that $\tilde{F}(L_N) \subset S(L_N \cap D)$ for $\tilde{F} \in \mathfrak{A}_c(L_N, Y)$ satisfying $\tilde{F}x \subset Fx$ for each $x \in L_N$. In fact, note that

$$\tilde{F}(L_N) \cap K \subset F(X) \cap K \subset S(N) \subset S(L_N \cap D).$$

On the other hand, $\tilde{F}(L_N) \backslash K \subset F(L_N) \backslash K \subset S(L_N \cap D)$ by (1.4). Therefore, we have $\tilde{F}(L_N) \subset S(L_N \cap D)$.

Note that $\tilde{F}(L_N)$ is compact since it is the image of the compact set L_N under $\tilde{F}$. Therefore, $\tilde{F}(L_N) \subset S(A)$ for some $A = \{x_0, x_1, \ldots, x_n\} \in \langle L_N \cap D \rangle$. Then, by Lemma, there exists an $f \in \mathbb{C}(\Delta_n, X)$ such that $f(\Delta_n) \subset \Gamma_A \cap L_N = \Gamma'_A$ and $f(\Delta_J) \subset \Gamma'_J$ for each $J \in \langle A \rangle$, where $\Delta_n = \operatorname{co}\{e_0, e_1, \ldots, e_n\}$ and Δ_J is the face of Δ_n corresponding to J. Let $\{\lambda_i\}_{i=0}^n$ be the partition of unity subordinated to the cover $\{Sx_i \cap \tilde{F}f(\Delta_n)\}_{i=0}^n$ of $\tilde{F}f(\Delta_n) \subset \tilde{F}(L_N)$.

For the remainder of the proof, we can just follow that of Case (i). This completes our proof.

REMARKS. 1. Theorem 1 for Case (ii) is a correct and generalized form of Ding [Di, Theorem 3.3]. As in [Di], Theorem 1 can be applied to intersection theorems concerning sets with H-convex sections and the von Neumann type minimax theorems.

2. Note that, if F is single-valued, the Hausdorffness assumption is not necessary. See [P1, Theorem 6], [P2, Theorem 4].

PARTICULAR FORMS. 1. If $(X, D; \Gamma)$ is a convex space with $\Gamma_N = \operatorname{co} N$ for each $N \in \langle D \rangle$, then Theorem 1 for Case (ii) includes Park [P5, Theorem 5], which was shown to be equivalent to a number of fundamental theorems in the KKM theory. For details, see [P5].

2. For H-spaces, Theorem 1 includes Horvath [H5, Theorem 4.2], Ding, Kim, and Tan [DKT1, Corollaries 3-5], Ding and Tan [DT, The-

orems 10-12 and Corollaries 2-4], Tarafdar [T, Theorem 2], Chen [Ch, Theorem 2], and Park [P1, Theorem 6], [P2, Theorem 4].

3. If F is compact in Theorem 1, by putting $\overline{F(X)} = K$, the coercivity condition (1.4) (ii) holds automatically. Therefore, we have the following:

COROLLARY 1. *Let $(X, D; \Gamma)$ be an H-space, Y a Hausdorff space, $F \in \mathfrak{A}_c^\kappa(X, Y)$ a compact map, and $S : D \multimap Y$, $T : X \multimap Y$. Suppose that*

(1) *for each $x \in D$, Sx is compactly open;*
(2) *for each $y \in F(X)$, $M \in \langle S^- y\rangle$ implies $\Gamma_M \subset T^- y$; and*
(3) *$\overline{F(X)} \subset S(D)$.*

Then F and T have a coincidence point $\bar{x} \in X$; that is, $F\bar{x} \cap T\bar{x} \neq \emptyset$.

REMARK. Condition (2) in Corollary 1 can be replaced by the following:

(2)′ for each $x \in D$, $Sx \subset Tx$ and, for each $y \in F(X)$, $T^- y$ is H-convex.

PARTICULAR FORMS. For H-spaces, Corollary 1 includes Horvath [H1, Théorème 4.1], [H2, Théorème 2 et Lemma 1], and [H3, Theorem 2′].

Moreover, from Corollary 1, we have

COROLLARY 2. *Let $(X, D; \Gamma)$ be an H-space, Y a Hausdorff space, $F \in \mathfrak{A}_c^\kappa(X, Y)$ compact, and $G : X \multimap Y$. Suppose that*

(1) *for each $y \in F(X)$, $G^- y$ is H-convex; and*
(2) *$\{\text{Int}\, Gx : x \in D\}$ covers $\overline{F(X)}$.*

Then F and G have a coincidence point.

Proof. Put $G = T$ and let $S : D \multimap Y$ be defined by $Sx = \text{Int}\, Gx$ for $x \in D$ in Corollary 1.

PARTICULAR FORMS. If $(X, D; \Gamma)$ is a convex space, then Corollary 2 reduces to Park [P5, Theorem 2]. For H-spaces, Corollary 2 includes Horvath [H4, Corollaire 6] and Ding and Tarafdar [DTr, Theorem 3.1].

4. The KKM theorems

In this section, we show that Theorem 1 is equivalent to the following KKM type theorem:

THEOREM 2. *Let $(X, D; \Gamma)$ be an H-space, Y a Hausdorff space, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Let $G : D \multimap Y$ be a map such that*

(2.1) *for each $x \in D$, Gx is compactly closed in Y;*
(2.2) *for each $N \in \langle D \rangle$, $F(\Gamma_N) \subset G(N)$; and*
(2.3) *there exists a nonempty compact subset K of Y such that either*
  (i) *$\bigcap\{Gx : x \in M\} \subset K$ for some $M \in \langle D \rangle$; or*
  (ii) *for each $N \in \langle D \rangle$, there exists a compact H-subspace L_N of X containing N such that*

$$F(L_N) \cap \bigcap\{Gx : x \in L_N \cap D\} \subset K.$$

Then we have

$$\overline{F(X)} \cap K \cap \bigcap\{Gx : x \in D\} \neq \emptyset.$$

Proof. Let $S : D \multimap Y$, $H : Y \multimap X$, and $T : X \multimap Y$ be defined by $Sx = Y \backslash Gx$ for $x \in D$, $Hy = \bigcup\{\Gamma_M : M \in \langle S^- y \rangle\}$ for $y \in Y$, and $Tx = H^- x$ for $x \in X$. Then (1.1) and (1.2) hold by (2.1) and the definitions of S and T. Suppose that $\overline{F(X)} \cap K \cap \bigcap\{Gx : x \in D\} = \emptyset$; that is, $\overline{F(X)} \cap K \subset S(D)$, which is just (1.3). Note that (2.3) is equivalent to (1.4). Therefore, by Theorem 1, T and F have a coincidence point $\bar{x} \in X$; that is, $F\bar{x} \cap T\bar{x} \neq \emptyset$. For $y \in F\bar{x} \cap T\bar{x}$, we have $\bar{x} \in T^- y$ and hence, there exists an $N \in \langle S^- y \rangle \subset \langle D \rangle$ such that $\bar{x} \in \Gamma_N$. Since $y \in F\bar{x} \subset F(\Gamma_N) \subset G(N) = Y \backslash \bigcap\{Sx : x \in N\}$, $y \notin Sx$ for some $x \in N$; that is, $x \notin S^- y$ and $x \in N$, which is a contradiction. This completes our proof.

Theorems 1 and 2 are equivalent:

Proof of Theorem 1 using Theorem 2. Let $Gx = Y \backslash Sx$ for $x \in D$. Then (2.1) and (2.3) follow from (1.1) and (1.4), respectively. Moreover, from (1.3), we have

$$\overline{F(X)} \cap K \subset S(D) = \bigcup_{x \in D} (Y \backslash Gx) = Y \backslash \bigcap_{x \in D} Gx,$$

contrary to the conclusion of Theorem 2. Therefore, F and G does not satisfy (2.2) and hence, there exist an $N \in \langle D \rangle$ and a $y \in F(\Gamma_N) \backslash G(N)$; that is, $y \notin Gx$ or $y \in Sx$ for all $x \in N$. Since $N \in \langle S^- y \rangle \subset \langle D \rangle$, we have $\Gamma_N \subset T^- y$ by (1.2). Since $y \in F(\Gamma_N)$, there exists an $\bar{x} \in \Gamma_N$ such that $y \in F\bar{x}$. Note that $\bar{x} \in \Gamma_N \subset T^- y$; that is, $y \in T\bar{x}$. This completes our proof.

PARTICULAR FORMS. 1. If $(X, D; \Gamma)$ is a convex space, then Theorem 2 reduces to Park [P5, Theorem 7] which includes many known generalizations of the KKM theorem.

2. For H-spaces, Theorem 2 generalizes Horvath [H1, Théorème 3.1 et Corollaire 3], [H3, I, Theorem 1 and Corollary 1], Bardaro and Ceppitelli [BC1, Theorem 1], Ding and Tan [DT, Corollary 1 and Theorem 8], Ding, Kim, and Tan [DKT1, Lemma 1], and Park [P1, Theorems 1 and 4], [P2, Theorem 1].

REMARKS. 1. If $(X, D; \Gamma)$ is a convex space with $\Gamma_A = \mathrm{co}\, A$ for $A \in \langle D \rangle$, then (i) implies (ii). In fact, we can choose $L_N = \mathrm{co}(M \cup N)$.

2. Ding [Di, Theorem 3.2] claimed that Theorem 2 for Case (ii) holds for a compact-valued u.s.c. map F and

(2.2)′ for each $N \in \langle D \rangle$, $\Gamma_N \subset F^- G(N)$

instead of (2.2). However, this is false as the following example shows:

EXAMPLE. Let $X = Y = K = [0, 1]$, $D = \{0, 1\}$, and $\Gamma_N = \mathrm{co}\, N$ for $N \in \langle D \rangle$. Let $Gx = \{x\}$ for $x \in D$ and $Fx = [0, 1]$ for $x \in X$. Then $\bigcap\{Gx : x \in D\} = \emptyset$ and for each $N \in \langle D \rangle$, $\mathrm{co}\, N \subset F^- G(N) = [0, 1]$.

5. Fixed point theorems

Our main coincidence theorem can be applied to some fixed point problems for the class $\mathfrak{A}_c$.

THEOREM 3. *Let $(X, D; \Gamma)$ be an H-space such that X has a Hausdorff uniform structure and $F \in \mathfrak{A}_c(X, X)$ a compact map. Suppose that, for each entourage V of X, there exist maps $S : D \multimap X$ and $T : X \multimap X$ satisfying (1)-(3) in Corollary 1 and $\mathrm{Gr}(F) \cap \mathrm{Gr}(T) \subset V$. Then F has a fixed point.*

Proof. For each entourage V of the Hausdorff uniformity, by Corollary 1, there exist a map $T : X \multimap X$ and a point (x_V, y_V) such that

$(x_V, y_V) \in \overline{\mathrm{Gr}(F)} \cap \mathrm{Gr}(T) \subset V$. Therefore, F has a V-fixed point. Since $\overline{F(X)}$ is compact and F is u.s.c., F must have a fixed point.

PARTICULAR FORMS. See Horvath [H5, Theorem 4.4] and Park [P3, Theorem 5].

For metric spaces, we have the following from Theorem 3:

THEOREM 4. *Let $(X, D; \Gamma)$ be a metric H-space with the metric d, and $F \in \mathfrak{A}_c(X, X)$ a compact map. Suppose that*
(4.1) *every open ball in X is H-convex; and*
(4.2) *D is dense in $\overline{F(X)}$.*
Then F has a fixed point.

Proof. Note that X is a Hausdorff uniform space. For any $\varepsilon > 0$ and an entourage $V_\varepsilon = \{(x,y) \in X \times X : d(x,y) < \varepsilon\}$, define $S : D \multimap X$ and $T : X \multimap X$ by

$$Sx = B(x, \varepsilon) = \{y \in X : d(x,y) < \varepsilon\} \quad \text{for} \quad x \in D,$$

and

$$Tx = B(x, \varepsilon) \quad \text{for} \quad x \in X.$$

Note that $Sx = Tx$ for $x \in D$ and Sx is open in X. Moreover, for each $y \in X$,

$$T^{-}y = \{x \in X : y \in B(x,\varepsilon)\} = B(y, \varepsilon)$$

is H-convex. Furthermore, $\overline{F(X)} \subset S(D)$ by (4.2). Note that

$$\mathrm{Gr}(T) = \{(x,y) \in X \times X : d(x,y) < \varepsilon\} = V_\varepsilon.$$

Therefore, by Theorem 3, F has a fixed point.

PARTICULAR FORMS. 1. For $\mathfrak{A} = \mathbb{C}$, Theorem 4 extends Brouwer [Br], Schauder [Sc], Rassias [R], and Park [P3, Theorems 3 and 6]. For details, see [P3].

2. For $\mathfrak{A} = \mathbb{K}$, Theorem 4 extends Kakutani [Ka] and Bohnenblust and Karlin [BK].

6. Open-valued KKM theorems and coincidence theorems

In order to obtain the open version of Theorem 2 for the class $\mathfrak{A}_c$, we need the following:

THEOREM 5. *Let* $(X, D; \Gamma)$ *be an* H*-space,* $D = \{x_0, x_1, \cdots, x_n\} \in \langle X\rangle$, Y *a regular space,* $G : D \multimap Y$ *a map, and* $F : X \multimap Y$ *a compact valued u.s.c. map. If* $G : D \multimap Y$ *is a open valued map such that*

(5.1) *for each* $J \in \langle D\rangle$, $F(\Gamma_J) \subset G(J)$.

Then there is a closed valued map $H : D \multimap Y$ *such that* $Hx \subset Gx$ *for all* $x \in D$ *and*

(5.2) $Ff(\Delta_J) \subset H(J)$ *for each* $J \in \langle D\rangle$ *and* $f : \Delta_n \to \Gamma_D$ *in Lemma, where* Δ_J *is the face of* Δ_n *corresponding to* J.

Proof. For any $y \in G(D)$, let

$$H_y = \bigcap\{Gx : y \in Gx\};$$

then H_y is an open set in Y containing y. By the regurality of Y, there exists an open neighborhood U_y of y in Y such that

$$y \in U_y \subset \overline{U_y} \subset H_y.$$

Clearly for any $J \in \langle D\rangle$, we have

$$G(J) = \{U_y : y \in G(J)\}$$

and so by (5.1), $\bigcup\{U_y : y \in G(J)\}$ is an open cover of $Ff(\Delta_J)$, since $Ff(\Delta_J) \subset F(\Gamma_J)$.

Since $Ff(\Delta_J)$ is compact, there exists a $B_J \in \langle G(J)\rangle$ such that

$$Ff(\Delta_J) \subset \bigcup\{U_y : y \in B_J\}.$$

Let $B = \bigcup\{B_J : J \in \langle D\rangle\}$. Define $H : D \multimap Y$ by

$$Hx = \bigcup\{\overline{U_y} : y \in B \cap Gx\}$$

for each $x \in D$. Then Hx is closed in Y for each $x \in D$ and $Hx \subset Gx$, since $\overline{U_y} \subset H_y \subset Gx$ if $y \in Gx$. And for each $J \in \langle D\rangle$ and $z \in Ff(\Delta_J)$, we have $z \in U_y$ for some $y \in B_J \subset G(J) \cap B$; that is $y \in Gx \cap B$ for some $x \in J$, hence

$$Ff(\Delta_J) \subset H(J).$$

This completes our proof.

REMARK. The proof was motivated by Shih [Sh, Theorem 1].

THEOREM 6. *Let $(X, D; \Gamma)$ be an H-space, $D = \{x_0, x_1, \cdots, x_n\} \in \langle X \rangle$, Y a T_1 regular space, $G : D \multimap Y$ a map, and $F \in \mathfrak{A}_c(X, Y)$. Suppose that*

(6.1) *for each $x \in D$, Gx is open in Y; and*
(6.2) *for each $N \in \langle D \rangle$, $F(\Gamma_N) \subset G(N)$.*

Then $F(\Gamma_D) \cap \bigcap\{Gx : x \in D\} \neq \emptyset$.

Proof. Suppose that $F(\Gamma_D) \cap \bigcap\{Gx : x \in D\} = \emptyset$. By Theorem 5, there exists a map $H : D \multimap Y$ such that $Hx \subset Gx$ for each $x \in D$ and (5.2) holds, and so $F(\Gamma_D) \cap \bigcap\{Hx : x \in D\} = \emptyset$. Then $Ff(\Delta_n) \subset T(D) = Y$, where $Tx = Y \backslash Hx$ for each $x \in D$. Let $\{\lambda_i\}_{i=0}^n$ be the partition of unity subordinate to this cover $\{Tz_i\}_{i=0}^n$ of compact subset $Ff(\Delta_n)$ of Y. Define $p : Ff(\Delta_n) \to \Delta_n$ by

$$py = \sum_{i=0}^{n} \lambda_i(y) e_i = \sum_{i \in N_y} \lambda_i(y) e_i$$

for $y \in Ff(\Delta_n)$ where $i \in N_y \iff \lambda_i(y) \neq \emptyset \implies y \in Tx_i$.

By Lemma, $pFf \in \mathfrak{A}_c(\Delta_n, \Delta_n)$ has a fixed point $z_0 \in \Delta_n$; that is $z_0 \in pFfz_0$. So there is a $y_0 \in Ffz_0$ such that $py_0 = z_0$ and $y_0 \in Fx_0$ where $x_0 = fz_0$. If $i \in N_{y_0}$, then $y_0 \in Tx_i$, and

$$y_0 \in Fx_0 \cap \bigcap\{Tx_i : i \in N_{y_0}\} \neq \emptyset.$$

Put $M = \{x_i : i \in N_{y_0}\}$, then $Ff(\Delta_M) \cap \bigcap\{Tx : x \in M\} \neq \emptyset$; that is, $Ff(\Delta_M) \not\subset H(M)$, where Δ_M is the face of Δ_n corresponding to M. This contradicts (5.2). This completes our proof.

REMARKS. 1. In the proof of Theorem 6, we actually showed the following:

Let X be a topological space, $f : \Delta_n \to X$ a continuous function, and Y, D, F and G be the same as in Theorem 6. Suppose that (6.1) and

(6.2)′ for each $N \subset D$, $Ff(\Delta_N) \subset G(N)$.

Then $Ff(\Delta_n) \cap \bigcap\{Gx : x \in D\} \neq \emptyset$.

2. If F is a continuous single-valued map, then the T_1 regularity of Y is not necessary. See Park [P1, Theorem 14].

PARTICULAR FORMS. For convex spaces, Theorem 6 extends W.K. Kim [Ki1-2], Lassonde [L2, Theorem 1], and Park [P4, Theorem 10].

By the same argument to that of Theorems 1 and 2, we can show that Theorem 6 is equivalent to the following:

THEOREM 7. *Let* $(X, D; \Gamma)$ *be an* H*-space,* $D \in \langle X \rangle$, Y *a* T_1 *regular space, and* $F \in \mathfrak{A}_c(X, Y)$. *Let* $S : D \multimap Y$ *and* $T : X \multimap Y$ *satisfy the following:*

(7.1) *for each* $x \in D$, Sx *is closed in* Y;
(7.2) *for each* $y \in F(X)$, $M \in \langle S^- y \rangle$ *implies* $\Gamma_M \subset T^- y$; *and*
(7.3) $F(\Gamma_D) \subset S(D)$.

Then T *and* F *have a coincidence point* $\bar{x} \in X$; *that is,* $T\bar{x} \cap F\bar{x} \neq \emptyset$.

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