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A HOPF BIFURCATION IN A PARABOLIC FREE BOUNDARY PROBLEM WITH PUSHCHINO DYNAMICS

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1. Introduction

A Hopf bifurcation of a free boundary (or an internal layer) occurs in solidification, chemical reactions and combustion. It is a well-known fact that a free boundary usually appear as sharp transitions with narrow width between two materials ([2]). These phenomena can be described by reaction diffusion systems with a small layer parameter ε and a controlling parameter τ

$$
(1) \qquad \begin{aligned} \varepsilon\tau u_t &= \varepsilon^2 u_{xx} + f(u,v) \\ v_t &= Dv_{xx} + g(u,v), \quad (x,t) \in (0,1) \times (0,\infty). \end{aligned}
$$

Here u and v measure the levels of two diffusing quantities. The functions u and v satisfy Neumann boundary conditions at $x = 0, 1$. The reaction terms are assumed to be of the bistable type which means that the nullcline of f and g have three intersection points and the curve $f = 0$ determines as a triple valued function of v. This system is a model of the time evolution of interaction between two separated population and also a model of the mixing of chemically reacting-diffusing substances.

When ε and τ are chosen to be very small, the system (1) models a situation in which the quantity measured by u reacts much faster than that measured by v (τ small), while at the same time u diffuses slower

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than v (ε small). The principal interest in systems like (1) comes from the fact that there exist families of stationary solutions parametrized by ε, which approach discontinuous functions of x as $\varepsilon \to 0$. When ε is small, the stationary solution, being smooth, exhibits an abrupt but continuously differentiable transition at the location of the limiting discontinuity. The transition takes place with in an x-interval of length $O(\varepsilon)$. An x-interval, in which such an abrupt change takes place, is loosely called a layer — a boundary layer when it is adjacent to an endpoint of the interval or an internal layer when it is in the interior of the interval.

In 1989, Nishiura and Mimura [5] showed that the stationary solutions of (1) loses stability and there is a Hopf bifurcation as a parameter τ varies (in this case $\varepsilon \neq 0$). We are interesting in an occurance of a Hopf bifurcation for the case $\varepsilon = 0$. Whenever the singular limit $\varepsilon \downarrow 0$ of the system (1), an analysis of the layer solutions suggests that the layer of width $O(\varepsilon)$ converges to an interfacial curve $x = s(t)$ in x, t-space as $\varepsilon \downarrow 0$. In 1992, J. Keener and A. Panfiliov used f and g are a piecewise-linear "Pushchino dynamics"[6] in order to show the wave evolution in heterogeneous excitable media of a cardiac tissue. The function f and g are given by $g(u,v) = kv - u$ and

$$\begin{aligned} f(u,v) =& u + c_1 v \quad for \quad u < u_-, \\ & -u + c_2 v - a \quad for \quad u_- < u < u_+, \\ & u + c_1(v-1) \quad for \quad u > u_+. \end{aligned}$$

where c_1, c_2, a and k are positive constants and u_-, u_+ are real numbers. By the bistable assumption, a constant k must satisfy $-c_1 < k < \frac{c_1(c_2-a)}{c_1+a}$.

When $\varepsilon = 0$, the problem (1) with applying these dynamics of f and g is reduced to the following free boundary problem

$$\begin{aligned} & v_t = v_{xx} - (c_1 + k)v + c_1 H(x - s(t)) \quad for\,(x,t) \in \Omega^- \cup \Omega^+, \\ & v_x(0,t) = 0 = v_x(1,t) \quad for\; t > 0, \\ & v(x,0) = v_0(x) \quad for\; 0 \le x \le 1, \\ & \tau \frac{ds}{dt} = C(v(s(t),t)) \quad for\; t > 0, \\ & s(0) = s_0, \; 0 < s_0 < 1, \end{aligned}$$

where $v(x,t)$ and $v_x(x,t)$ are assumed continuous in $\Omega = (0,1)\times(0,\infty)$. Here, the function $H(\cdot)$ is the Heaviside function, $\Omega^- = \{(x,t) \in \Omega : 0 < x < s(t)\}$ and $\Omega^+ = \{(x,t) \in \Omega : s(t) < x < 1\}$.

In this paper, we will show the occurrence of a Hopf bifurcation as $\tau \downarrow 0$ in the free boundary problem (2). The velocity of the interface, $C(v)$, in (2), which specifies the evolution of the interface $s(t)$, is determined from the first equation in (2) using asymptotic techniques (see in [2], [4]). The function $C(v)$ can be calculated explicitly as

$$C(v) = \frac{2v - \frac{c_1 - 2a}{c_1 + c_2}}{\sqrt{(\frac{c_1 - a}{c_1 + c_2} - v)(v + \frac{a}{c_1 + c_2})}}.$$

In section 2, we introduce a change of variables to regularize problem (2). From this, we give an alternative proof of well-posedness and obtain enough regularity of the solution for an analysis of the bifurcation. In section 3, we show that as τ decreases, the stationay solutions lose stability which is results from a Hopf bifurcation and produces a kind of periodic oscillation in the location of internal layers.

2. Regularization

In this section, we obtain more regularity for the solution by semigroup methods since the nonlinear term of (2), $H(\cdot - s)$, is not differentiable. We write (2) as an abstract evolution equation

$$(F)\begin{cases} \frac{d(v,s)}{dt} + \widetilde{A}(v,s) = F(v,s), \\ (v,s)(0) = (v_0(\cdot), s_0). \end{cases}$$

of a differential equation in a space $\widetilde{X}$ of the form $\widetilde{X} = X \times \mathbf{J}$, where X is a Banach space of functions and J is a real interval. Here the operator $\widetilde{A}$ is 2×2 matrix

$$\widetilde{A} := \begin{pmatrix} -\frac{d^2}{dx^2} + (c_1 + k) & 0 \\ 0 & 0 \end{pmatrix}.$$

and the nonlinear operator F is

$$F(v,s) = \begin{pmatrix} F_1(v(\cdot,t), s(t)) \\ F_2(v(\cdot,t), s(t)) \end{pmatrix} := \begin{pmatrix} c_1 H(\cdot - s(t)) \\ \frac{1}{\tau} C(v(s(t),t)) \end{pmatrix}$$

The Neumann boundary conditions are incorporated in the definition of the Banach space X.

We consider a differential operator, $-\frac{d^2}{dx^2} + (c_1 + k)$ as a densely defined operator

$$\begin{aligned} Av &:= -v_{xx} + (c_1 + k)v \quad \text{with } v_x(0) = v_x(1) = 0 \\ &\qquad A : D(A) \subset_{\text{dense}} X \longrightarrow X \\ D(A) &:= \{v \in H^{2,2}((0,1)) \,:\, v_x(0) = v_x(1) = 0\} \end{aligned}$$

where

$$X := L_2((0,1)) \text{ with norm } \|\cdot\|_2.$$

For fixed s, the map $t \mapsto H(\cdot - s(t))$ is locally Hölder-continuous into X on $(0,T)$, so by standard results for parabolic problems (see e.g.[3]) we obtain from the first equation in (F) that the following regularity holds for v.

PROPOSITION 2.1. *If (v,s) is a solution of (F) then $v(\cdot,t) \in D(A)$ and the map $t \mapsto v(\cdot,t)$ is in $C^0([0,T),X) \cap C^1((0,T),X)$.*

Proof. Using the similar argument in [7], we obtain the above results. □

We define

$$g(x,s) = \int_0^1 c_1 \cdot G(x,y) \cdot H(y-s)\,dy = A^{-1}(c_1 H(\cdot - s))$$

and

$$\gamma(s) := g(s,s).$$

Let

$$u(t)(x) := v(x,t) - g(x,s(t)).$$

We choose the space $X \times \mathbf{R}$ by $\widetilde{X}$ and define

$$\begin{aligned} &D(\widetilde{A}) := D(A) \times \mathbf{R}, \\ &\widetilde{A} : D(\widetilde{A}) \subset_{\text{dense}} \widetilde{X} \longrightarrow \widetilde{X}, : \widetilde{A}(u,s) := (Au, 0). \end{aligned}$$

The corresponding evolution system to the regular problem of (1) with an initial value problem for (u, s) can then be written as

$$(R)\begin{cases} \frac{d}{dt}(u,s) + \widetilde{A}(u,s) = \dfrac{1}{\tau} f(u,s) \\ (u,s)(0) = (u(0), s(0)) = (u_0, s_0) \,. \end{cases}$$

Here a nonlinear forcing term f is defined on the set

$$W := \{(u,s) \in C^1([0,1]) \times (0,1) \,:\, u(s) + \gamma(s) \in I\} \subset C^1([0,1]) \times \mathbf{R}$$

and

$$f : W \to X \times \mathbf{R}, \quad f(u,s) := f_2(u,s) \cdot (f_1(s), 1)$$

where

$$f_1 : (0,1) \to X, \quad f_1(s)(x) := G(x,s)$$
$$f_2 : W \to \mathbf{R}, \quad f_2(u,s) := C(u(s) + \gamma(s)).$$

Then we can show the regularity of f.

LEMMA 2.2. *The functions $f_1 : (0,1) \to X$, $f_2 : W \to \mathbf{R}$ and $f : W \to \widetilde{X}$ are continuously differentiable with derivatives given by*

$$f_1'(s) = \frac{\partial G}{\partial y}(\cdot, s)$$
$$Df_2(u,s)(\hat{u}, \hat{s}) = C'(u(s) + \gamma(s)) \cdot (u'(s)\hat{s} + \gamma'(s)\hat{s} + \hat{u}(s))$$
$$Df(u,s)(\hat{u}, \hat{s}) = f_2(u,s) \cdot (f_1'(s), 0) \cdot \hat{s} + Df_2(u,s)(\hat{u}, \hat{s}) \cdot (f_1(s), 1) \,.$$

Proof. The proof is similar to the Lemma 2.4 in [6]. □

We now apply semigroup theory to (R) using domains of fractional powers $\alpha \in [0,1]$ of A and $\widetilde{A}$:

$$X^\alpha := D(A^\alpha)\,, \quad \widetilde{X}^\alpha := D(\widetilde{A}^\alpha)\,, \quad \widetilde{X}^\alpha = X^\alpha \times \mathbf{R}\,.$$

For this we need to find an $\alpha \in (0,1)$ such that $X^\alpha \subset C^1([0,1])$, because then $f : W \cap \widetilde{X}^\alpha \to \widetilde{X}$ is continuously differentiable. By the imbedding theorem in [3], we have the following wellposedness result.

THEOREM 2.3.

(i) *For any* $1 > \alpha > 3/4$, $(u_0, s_0) \in W \cap \widetilde{X}^\alpha$ *and* $\tau \in \mathbf{R}$ *there exists a unique solution*

$$(u,s)(t) = (u,s)(t; u_0, s_0, \tau)$$

of (R). The solution operator

$$(u_0, s_0, \tau) \mapsto (u,s)(t; u_0, s_0, \tau)$$

is continuously differentiable from $\widetilde{X}^\alpha \times \mathbf{R}$ *into* $\widetilde{X}^\alpha$ *for* $t > 0$. *The functions* $v(x,t)$

$$v(x,t) := u(t)(x) + g(x, s(t))$$

and s *then satisfy (F) with* $v(\cdot, 0) \in X^\alpha$, $v(s_0, 0) \in I$.

(ii) *If* (v,s) *is a solution of (F) for some* $\mu \in \mathbf{R}$ *with initial condition* $v_0 \in X^\alpha$, $1 > \alpha > 3/4$, $s_0 \in (0,1)$, $v_0(s_0) \in I$, *then* $(u_0, s_0) := (v_0 - g(\cdot, x_0), s_0) \in \widetilde{X}^\alpha \cap W$ *and*

$$(v(\cdot,t), s(t)) = (u,s)(t; u_0, s_0, \tau) + (g(\cdot, s(t)), 0)$$

where $(u,s)(t; u_0, s_0, \tau)$ *is the unique solution of (R).*

(iii) *For any* $1 > \alpha > 3/4$, $\mu \in \mathbf{R}$, $(v_0, s_0) \in U := \{(v,s) \in X^\alpha \times (0,1) : v(s) \in I\}$ *the problem (F) has a unique solution*

$$(v(x,t), s(t)) = (v,s)(x,t; v_0, s_0, \tau).$$

Additionally, the mapping

$$(v_0, s_0, \tau) \mapsto (v,s)(\cdot, t; v_0, s_0, \tau)$$

is continuously differentiable from $X^\alpha \times \mathbf{R}^2$ *into* $X^\alpha \times \mathbf{R}$.

3. Stationary solutions and Hopf bifurcation

3.1 Stationary solutions

We introduce a new parameter $\mu \in R^+$, $\mu = \dfrac{2(c_1 + c_2)}{\tau c_1}$ and $c^2 = c_1 + k$. The function $\gamma(s)$ which is defined in the section 2 becomes

$$\begin{aligned}\gamma(s) &= \int_s^1 c_1 G(s, y)\, dy \\ &= \frac{c_1}{2c^2}\left(1 - \frac{\sinh(c(2s-1))}{\sinh c}\right)\end{aligned}$$

and we have

$$\gamma'(s) < 0, : \gamma(0) = \frac{1}{c^2}, : \gamma(1) = 0. \tag{3}$$

We thus obtain the existence of stationary solutions of (R).

PROPOSITION 3.1. *If* $0 < \dfrac{c_1 - 2a}{2(c_1 + c_2)} < \dfrac{1}{(c_1 + k)}$ *then (R) has a unique stationary solution* $(0, s^*)$ *for all* $\mu \neq 0$ *with* $s^* \in (0,1)$. *The linearization of* f *at* $(0, s^*)$ *is*

$$Df(0, s^*)(\hat{u}, \hat{s}) = \Big(\hat{u}(s^*) + \gamma'(s^*)\hat{s}\Big) \cdot \Big(f_1(s^*), 1\Big).$$

The pair $(0, s^*)$ *corresponds to a unique steady state* (v^*, s^*) *of (F) for* $\mu \neq 0$ *with*

$$v^*(x) = g(x, s^*).$$

Proof. Since $C(r) = 0$ iff $r = \dfrac{c_1 - 2a}{2(c_1 + c_2)}$, the stationary problem is solvable with $s^* \in (0,1)$ iff $\gamma(0) > \dfrac{c_1 - 2a}{2(c_1 + c_2)} > \gamma(1)$ (see (3)), which means $\dfrac{1}{(c_1 + k)} > \dfrac{c_1 - 2a}{2(c_1 + c_2)} > 0$.

The formula for $Df(0, s^*)$ follows from Lemma 2.2 and the relation $C'\left(\dfrac{c_1 - 2a}{2(c_1 + c_2)}\right) = \dfrac{2(c_1 + c_2)}{c_1}$. The corresponding steady state (v^*, s^*) for (F) is obtained using Theorem 2.3. □

3.2 A Hopf bifurcation

We next want to show that there is a Hopf bifurcation from the curve $\mu \mapsto (0, s^*)$ of steady states and therefore introduce the following definition

DEFINITION 3.2. *Under the assumptions of Proposition 3.1, define (for $1 \geq \alpha > 3/4$) the operator $B \in L(\widetilde{X}^\alpha, \widetilde{X})$,*

$$B := Df(0, s^*).$$

We then define $(0, s^*, \mu^*)$ to be a Hopf poit for (R) if and only if there exists an $\varepsilon_0 > 0$ and a C^1- curve

$$(-\varepsilon_0 + \mu^*, \mu^* + \varepsilon_0) \mapsto (\lambda(\mu), \phi(\mu)) \in \mathbf{C} \times \widetilde{X}_{\mathbf{C}}$$

($Y_{\mathbf{C}}$ denotes the complexification of the real space Y) of a pair of eigenvalue and corresponding eigenfunction, so called eigendata for $-\widetilde{A}+\mu B$ with

(i) $(-\widetilde{A} + \mu B)(\phi(\mu)) = \lambda(\mu)\phi(\mu), \quad (-\widetilde{A} + \mu B)(\overline{\phi(\mu)}) = \overline{\lambda(\mu)}\,\overline{\phi(\mu)}$;
(ii) $\lambda(\mu^*) = i\beta$ *with* $\beta > 0$;
(iii) $Re(\lambda) \neq 0$ *for all* $\lambda \in \sigma(-\widetilde{A} + \mu^* B) \setminus \{\pm i\beta\}$;
(iv) $Re\,\lambda'(\mu^*) \neq 0$ *(transversality).*

A Hopf point $(0, s^*, \mu^*)$ is the origin of a C^0-curve of initial conditions (u_0, s_0) for nontrivial periodic solutions. This basically follows from a Theorem in [1], but the proof requires a little reinvestigation, for the theorem is only stated for C^2-nonlinearities f and then yields a C^1-curve of bifurcating periodic orbits. Since we are unable to meet the C^2 requirement, we indicate briefly how to modify the proof, using an implicit function theorem that only requires differentiability with respect to one part of the arguments.

THEOREM 3.3. [Hopf-Bifurcation] *Assume $(0, s^*, \mu^*)$ is a Hopf point for (R). Then there exists $\epsilon_1 > 0$ and a C^0-curve*

$$\epsilon \in (-\epsilon_1, \epsilon_1) \mapsto (u_0(\epsilon), s_0(\epsilon), p(\epsilon), \mu(\epsilon)) \in \widetilde{X}^\alpha \times \mathbf{R}^+ \times \mathbf{R}$$

such that

$$(u, s)(\cdot; u_0(\epsilon), s_0(\epsilon), \mu(\epsilon))$$

is a periodic solution of (R) with (primitive) period $p(\epsilon)$.

Moreover $u_0(0) = 0, : s_0(0) = s^*, : p(0) = \frac{2\pi}{\beta}, : \mu(0) = \mu^*$ *and*

$$\lim_{\epsilon \to 0} \frac{(u_0(\epsilon), s_0(\epsilon) - s^*)}{\epsilon} = Re\,\phi(\mu^*)\,.$$

Proof. Using the similar arguments in the proof of Theorem 3.3 which is in [7], we obtain the above results. □

We now have to check (R) for Hopf points. For this we have to solve the eigenvalue problem

$$-\widetilde{A}(u,s) + \mu B(u,s) = \lambda(u,s)$$

which by Proposition 3.1 is equivalent to

$$(4) \qquad \begin{aligned} (A+\lambda)u &= \mu \cdot \big(\gamma'(s^*)s + u(s^*)\big) \cdot G(\cdot, s^*) \\ \lambda s &= \mu \cdot \big(\gamma'(s^*) + u(s^*)\big). \end{aligned}$$

As a first result, we obtain that it suffices to find a unique, purely imaginary eigenvalue $\lambda = i\beta$ of (4) with $\beta > 0$ for some μ^* in order for $(0, s^*, \mu^*)$ to be a Hopf point.

THEOREM 3.4. *Assume that for* $\mu^* \in \mathbf{R} \setminus \{0\}$ *the operator* $-\widetilde{A} + \mu^* B$ *has a unique pair* $\{\pm i\beta\}$ *of purely imaginary eigenvalues. Then* $(0, s^*, \mu^*)$ *is a Hopf point for (R).*

Proof. Without loss of generality, let $\beta > 0$, and let ϕ^* be the (normalized) eigenfunction of $-\widetilde{A} + \mu^* B$ with eigenvalue $i\beta$. We have to show that $(\phi^*, i\beta)$ can be extended to a C^1-curve $\mu \mapsto (\phi(\mu), \lambda(\mu))$ of eigendata for $-\widetilde{A} + \mu B$ with $\lambda'(\mu^*) \neq 0$.

For this let $\phi^* = (\psi_0, s_0) \in D(A) \times \mathbf{R}$. First, we see that $s_0 \neq 0$, for otherwise, by (4), $(A + i\beta)\psi_0 = i\beta s_0 G(\cdot, s^*) = 0$, which is not possible because A is symmetric. So without loss of generality, let $s_0 = 1$. Then by (4) $E(\psi_0, i\beta, \mu^*) = 0$, where

$$E : D(A)_{\mathbf{C}} \times \mathbf{C} \times \mathbf{R} \longrightarrow X_{\mathbf{C}} \times \mathbf{C}$$

and

$$E(u,\lambda,\mu)$$
$$:= \Big((A+\lambda)u - \mu\cdot(\gamma'(s^*)+u(s^*))G(\cdot,s^*), \lambda - \mu\cdot(\gamma'(s^*)+u(s^*))\Big).$$

The equation $E(u,\lambda,\mu) = 0$ is equivalent that λ is an eigenvalue of $-\widetilde{A} + \mu B$ with eigenfunction $(u,1)$. We want to apply the implicit function theorem to E, and therefore have to check that E is in C^1 and that

(5) $D_{(u,\lambda)}E(\psi_0, i\beta, \mu_0) \in L(D(A)_{\mathbf{C}} \times \mathbf{C}, X_{\mathbf{C}} \times \mathbf{C})$ is an isomorphism.

Now it is easy to see that

$$\begin{aligned} & D_uE(u,\lambda,\mu)\hat{u} = \Big((A+\lambda)\hat{u} - \mu\hat{u}(s^*)G(\cdot,s^*), -\mu\hat{u}(s^*)\Big) \\ (6) \qquad & D_\lambda E(u,\lambda,\mu)\hat{\lambda} = \hat{\lambda}(u,1) \\ & D_\mu E(u,\lambda,\mu)\hat{\mu} = -\hat{\mu}(\gamma'(s^*)+u(s^*))\cdot(G(\cdot,s^*),1) \end{aligned}$$

so E is C^1. In addition, the mapping

$$D_{(u,\lambda)}E(\psi_0, i\beta, \mu^*)(\hat{u},\hat{\lambda})$$
$$= \Big((A+i\beta)\hat{u} - \mu^*\hat{u}(s^*)\cdot G(\cdot,s^*) + \hat{\lambda}\psi_0, -\mu^*\hat{u}(s^*) + \hat{\lambda}\Big)$$

is a compact perturbation of the mapping

$$(\hat{u},\hat{\lambda}) \longmapsto \Big((A+i\beta)\hat{u}, \hat{\lambda}\Big)$$

which is invertible. As a consequence, $D_{(u,\lambda)}E(\psi_0, i\beta, \mu^*)$ is a Fredholm operator of index 0. Thus to verify (5), it suffices to show that the system

$$\begin{aligned} & (A+i\beta)\hat{u} + \hat{\lambda}\psi_0 = \mu^*\hat{u}(s^*)G(\cdot,s^*) \\ (7) \qquad & \hat{\lambda} = \mu^*\hat{u}(s^*) \end{aligned}$$

necessarily implies that $\hat{u} = 0$, $\hat{\lambda} = 0$. Thus let $(\hat{u}, \hat{\lambda})$ be a solution of (7), and define $\psi_1 := \psi_0 - G(\cdot, s^*)$. Then

$$(A + i\beta)\hat{u} + \hat{\lambda}\psi_1 = 0 \tag{8}$$

On the other hand, since ψ_0 solves (4) with $\lambda = i\beta$ and $s = 1$, we have

$$i\beta G(\cdot, s^*) = A\psi_0 + i\beta\psi_0 = A\psi_1 + \delta_{s^*} + i\beta G(\cdot, s^*)$$

in the weak sense. Here δ_s is the delta-distribution centered at s. So ψ_1 is a solution to the equation

$$(A + i\beta)\psi_1 = -\delta_{s^*} \tag{9}$$

and

$$i\beta = \mu^* \cdot \Big(\gamma'(s^*) + \psi_0(s^*)\Big) = \mu^* \cdot \Big(\gamma'(s^*) + \psi_1(s^*) + G(s^*, s^*)\Big) . \tag{10}$$

Equation (9) implies that

$$-\overline{\psi_1(s^*)} = \int_0^1 |A^{1/2}\psi_1|^2 + i\beta \int_0^1 |\psi_1|^2 ,$$

so that

$$\operatorname{Im} \psi_1(s^*) = \beta \int_0^1 |\psi_1|^2 .$$

Now $\gamma'(s^*)$ and $G(s^*, s^*)$ in (10) are real valued, therefore, since $\beta \neq 0$

$$\mu^* \int_0^1 |\psi_1|^2 = 1 . \tag{11}$$

From (9) we can then calculate $\hat{u}(s^*)$ as $\int_0^1 \psi_1(A + i\beta)\hat{u} = -\hat{u}(s^*)$, which together with (8), (9) and (11) implies that

$$\hat{\lambda} \int_0^1 \psi_1^2 = \hat{u}(s^*) = \hat{\lambda}/\mu^* = \hat{\lambda} \int_0^1 |\psi_1|^2 .$$

As a result

$$\hat{\lambda}\left(\int_0^1 |\psi_1|^2 - \psi_1^2\right) = 0\,,$$

which implies $\hat{\lambda} = 0$, for otherwise $\operatorname{Im}\psi_1 = \operatorname{Im}\psi_0 = 0$, which is a contradiction. So we conclude that $\hat{\lambda} = 0$, and with this that also $\hat{u} = 0$.

We have thus shown (5), and therefore get a C^1-curve $\mu \mapsto (\phi(\mu), \lambda(\mu))$ of eigendata such that $\phi(\mu^*) = \phi^*$ and $\lambda(\mu^*) = i\beta$. It remains to be shown that $\operatorname{Re}\lambda'(\mu^*) \neq 0$. Let $\phi(\mu) = (\psi(\mu), 1)$. Implicit differentiation of $E(\psi(\mu), \lambda(\mu), \mu) = 0$ (see (6)) implies that

$$\begin{aligned} &D_{(u,\lambda)}E(\psi_0, i\beta, \mu^*)(\psi'(\mu^*), \lambda'(\mu^*)) \\ &= \Big(\gamma'(s^*) + \psi'(\mu^*)(s^*)\Big) \cdot \Big(G(\cdot, s^*), 1\Big)\,. \end{aligned}$$

This means that the function $\hat{u} := \psi'(\mu^*)$ and $\hat{\lambda} := \lambda'(\mu^*)$ satisfy the equations

$$(12) \qquad (A + i\beta)\hat{u} - \mu^*\hat{u}(s^*)G(\cdot, s^*) + \hat{\lambda}\psi_0 = (\gamma'(s^*) + \hat{u}(s^*))G(\cdot, s^*)$$

and

$$(13) \qquad \mu^*\hat{u}(s^*) + \hat{\lambda} = \gamma'(s^*) + \hat{u}(s^*)\,.$$

Putting (13) into (12) and using $\psi_1 := \psi_0 - G(\cdot, s^*)$, as before, we obtain

$$(A + i\beta)\hat{u} + \hat{\lambda}\psi_1 = 0\,,$$

and from here with (9) that

$$-\overline{\hat{u}(s^*)} = \int_0^1 (A + i\beta)\psi_1\overline{\hat{u}} = \int_0^1 \psi_1\overline{(A + i\beta)\hat{u}} = -\overline{\hat{\lambda}}\int_0^1 |\psi_1|^2 = -\overline{\hat{\lambda}}\frac{1}{\mu^*}\,,$$

where we have used (11) for the last step. We thus obtain $\hat{\lambda} = \mu^*\hat{u}(s^*)$ and from (13) that

$$\hat{\lambda} = \operatorname{Re}\hat{\lambda} = -\mu^*\gamma'(s^*) > 0\,.$$

□

We now need the following lemma in order to show the uniqueness of μ^*.

LEMMA 3.5. *Let G_β be Green's function for the operator $A + i\beta$. Then the expression $Re\, G_\beta(s^*, s^*)$ is strictly decreasing in $\beta \in \mathbf{R}^+$ with*

$$Re\, G_0(s^*, s^*) = G(s^*, s^*), \quad \lim_{\beta \to \infty} Re\, G_\beta(s^*, s^*) = 0\,,$$

and $Im\, G_\beta(s^*, s^*) < 0$ for any $\beta > 0$.

Proof. The argument of the proof is similar to the Lemma 3.5 in [7]. □

Therefore, we have the following result.

THEOREM 3.6. *Whenever (R) admits a stationary solution, there is a unique $\mu^* > 0$ such that $(0, s^*, \mu^*)$ is a Hopf point.*

Proof. We have only to show that the function from (u, β, μ) to $E(u, i\beta, \mu)$ has a unique zero with $\beta > 0$ and $\mu > 0$. This means solving the system

$$(A + i\beta)u = \mu \cdot \Big(\gamma'(s^*) + u(s^*)\Big) \cdot G(\cdot, s^*)$$
$$i\beta = \mu \cdot \Big(\gamma'(s^*) + u(s^*)\Big)\,.$$

As before, with $v := u - G(\cdot, s^*)$, this system is equivalent to the weak system of equations

$$(A + i\beta)v = -\delta_{s^*}$$
$$i\beta = \mu \cdot \Big(\gamma'(s^*) + G(s^*, s^*) + v(s^*)\Big)\,. \tag{14}$$

Now the first equation in (14) has, for fixed $\beta \geq 0$, the unique solution $v = -G_\beta(\cdot, s^*)$. We are thus left with having to solve the complex valued equation

$$i\beta = \mu \cdot (\gamma'(s^*) + G(s^*, s^*) - G_\beta(s^*, s^*))\,.$$

Since $\gamma'(s^*) + G(s^*, s^*)$ is real valued, this is equivalent to the real valued system

$$\gamma(s^*) + G(s^*, s^*) - \mathrm{Re}\, G_\beta(s^*, s^*) = 0 \tag{15}$$

$$\mu \cdot \mathrm{Im}\, G_\beta(s^*, s^*) + \beta = 0\,. \tag{16}$$

By $\gamma'(s^*) < 0$, $\gamma'(s^*) + G(s^*, s^*) > 0$ and Lemma 3.5, the existence of a unique solution (β, μ^*) of (15) and (16) with $\beta > 0$ and $\mu^* > 0$ follows by an application of the mean value theorem. $\square$

The following theorem summarizes what we have proved:

THEOREM 3.7. *Assume that* $0 < \dfrac{c_1 - 2a}{2(c_1 + c_2)} < \dfrac{1}{(c_1 + k)}$, *so that (R), respectively (F), has a unique stationary solution* $(0, s^*)$, *respectively* (v^*, s^*), *for all* $\mu > 0$. *Then there exists a unique* $\mu^* > 0$ *such that the linearization* $-\widetilde{A} + \mu^* B$ *has a purely imaginary pair of eigenvalues. The point* $(0, s^*, \mu^*)$ *is then a Hopf point for (R) and there exists a* C^0*-curve of nontrivial periodic orbits for (R), (F), respectively, bifurcating from* $(0, s^*, \mu^*)$, (v^*, s^*, μ^*), *respectively.*

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