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A DIFFUSION APPROXIMATION FOR TIME–DEPENDENT QUEUE SIZE DISTRIBUTION FOR $M/G/m/N$ SYSTEM

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1. Introduction

The purpose of this paper is to provide a transient diffusion approximation of queue size distribution for $M/G/m/N$ system. The $M/G/m/N$ system can be expressed as follows. The interarrival times of customers are exponential and the service times of customers have general distribution. The system can hold at most a total of N customers (including the customers in service) and any further arriving customers will be refused entry to the system and will depart immediately without service. The queueing system with finite capacity is more practical model than queueing system with infinite capacity. For example, in the design of a computer system one of the important problems is how much capacity is required for a buffer memory. If its capacity is too little, then overflow of customers (jobs) occurs frequently in heavy traffic and the performance of system deteriorates rapidly. On the other hand, if its capacity is too large, then most buffer memories remain unused.

The $M/G/m/N$ system is an important queueing model in its own right, moreover the model has an additional importance because of its equivalence to a t wo-stage cyclic queue as follows. Consider the cyclic queueing system of fig.1 (a) with a fixed number N of customers circulating the system, where the stage 1 has exponentially distributed service time with mean $\frac{1}{\lambda}$ and m servers in stage 2 has general service

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distribution. Then we can observe that the number of customers at stage 2 for two-stage cyclic queue is essentially same as the number of customers for $M/G/m/N$ queue. Customers arrive at stage 2 according to a Poisson stream with rate λ as long as stage 1 is busy. If stage 1 becomes idle, then all N customers are in stage 2, and there will be no arrivals to stage 2. This situation corresponds exactly to the case in which the equivalent $M/G/m/N$ queue has all the waiting rooms full.

For a single server system with finite capacity, the stationary queue size distribution has been studied by many authors (eg. Cohen [4] and references in it). However the exact results of queue size distribution for multiserver system with finite waiting room are not known except for some special cases. Thus our methods for performance on $M/G/m/N$ system are either to restrict the class of service time distributions or to develop tractable approximate method.

There are many practical situations where we need to know the transient behavior of queueing system (eg. Kobayashi [20], Duda [9]). One of the goals of time-dependent analysis of queueing systems is to characterize the transient behavor of system for various initial conditions. Difficulty in obtaining the exact solution of transient queue size distribution for $M/G/m/N$ system is quite apparent, and thus we shall have to content ourselves with approximate solution. The diffusion approximation may be useful in this regard.

Diffusion approximations for the various characteristics (stationary queue size distrbution, loss probability, first overflow time) in $GI/G/1/N$ system were derived by Kimura et al. [16, 17]. In the single server case, the transient approximations were investigated by using diffusion process with reflecting boundary for the heavy traffic case (Kobayashi [20], Abate and Whitt [1, 2]). A transient diffusion approximation for the queue size distribution in $M/G/m$ system with infinite capacity was obtained for all traffic cases by Choi and Shin [6].

In this paper we use an elementary return process with elementary return boundaries at $x = 0$ and $x = N$ to obtain an approximation of the transient queue size distribution in the $M/G/m/N$ system. In section 2, we describe the model precisely and derive the Laplace transform solution of diffusion equation to approximate the queue size distribution in the $M/G/m/N$ system. A stationary approximation is also given. In section 3, the accuracy of approximation is investigated by numerical comparison with simulation.

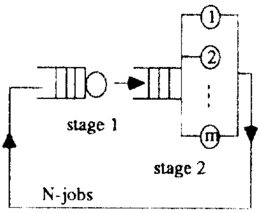

Fig. 1(a) 2-stage cyclic queueing system

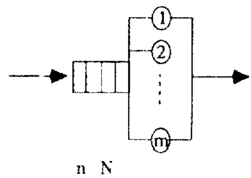

Fig. 2(b) $M/G/m/N$ system

2. Diffusion approximation for $M/G/m/N$ system

Consider an $M/G/m/N$ queueing system for which it is assumed that the system capacity is N (including m servers in service) and there are m identical servers acting in parallel. The customers arrive at the system according to a Poisson process with rate λ. The service time of each customer has general distribution with mean $\frac{1}{\mu}$ and variance σ_s^2, which are independent of the interarrival times and the number of customers in the system.

A basic idea in diffusion approxiamtion is to use the process with continuous state space which is mathematically tractable to approximate the process with discrete state space. The procedure for using the diffusion approximation method to approximate a queueing process can be summarized as following steps;

(i) Determine the state space, diffusion parameters and boundary conditions of diffusion process from the approximate means and variances of the inter-arrival time and service times in the queueing context and then obtain the Kolmogorov forward equation (or Fokker - Planck equation).

(ii) Solve the Fokker - Planck equation.

(iii) Select the appropriate discretization of the resulting probability density function to obtain the discrete probability mass function with state space $\{0, 1, 2, \cdots, N\}$.

Let $\{X(t), t \geq 0\}$ be a diffusion process which approximates the process $\{Y(t), t \geq 0\}$ representing the queue size in the $M/G/m/N$ system at time t. The lower and upper boundaries of the state space of $X(t)$ correspond to the zero state and the system capacity N in $M/G/m/N$

system, respectively. Thus the holding times at the lower and upper boundaries of process $\{X(t), t \geq 0\}$ correspond to the idle period of queueing system and the time period during which the system is full, respectively. It is easy to see that if the arrival process is Poisson, then the idle period has exponential distribution with mean $\frac{1}{\lambda}$. Thus the holding time distribution of $X(t)$ at the lower boundary is exponential with parameter λ. In the $M/G/m/N$ system, the exact distribution of the time period during which the system is full is not known. Hence the holding time of diffusion process at the upper boundary should be approximated. Since the set of all Cox distributions is dense in the set of probability distributions on $(0, \infty)$, that is, to any probability distribution F on $(0, \infty)$, there is a sequence $\{F_k\}$ of Cox distributions with $F_k \xrightarrow{w} F$ (eg. see Asmusen [3]), first we derive the transient density function of diffusion process when the holding time at the upper boundary has Cox distribution and then by continuity theorem we obtain the transient density function of the process with general distribution of holding time. The Laplace transform $h^*(s)$ of the density function $h(t)$ of Cox distribution is

$$h^*(s) = \sum_{i=1}^{n} d_i(1 - c_i)e_i^*(s)$$

where $e_i^*(s) = \prod_{j=1}^{i} \frac{\mu_i}{s+\mu_j}$ denotes the Laplace transform of the i−stage Erlang distribution and

$$d_i = \begin{cases} 1, & \text{if } i = 1 \\ c_1c_2\cdots c_{i-1}, & \text{if } i > 1. \end{cases}$$

The mean Λ of n-stage Cox distribution is

$$\Lambda = -\frac{dh^*(s)}{ds}\bigg|_{s=0} = \sum_{i=1}^{n} \frac{c_i}{\mu_i}.$$

The behavior of the process $X(t)$ in the interval $(0, N)$ is specified by the diffusion parameters $a(x)$ and $b(x)$, called the infinitesimal variance and infinitesimal mean defined by

$$a(x) = \lim_{\Delta t \to 0} \frac{Var(X(t + \Delta t) - X(t)|X(t) = x)}{\Delta t}$$

$$b(x) = \lim_{\Delta t \to 0} \frac{E(X(t + \Delta t) - X(t)|X(t) = x)}{\Delta t}.$$

In the single server case the diffusion parameters $a(x)$ and $b(x)$ are determined by the view of central limit theorem (Kobayashi [21]). In the multiple server case the following modified formulas

$$a(x) = \lambda + \min(\lceil x \rceil, m)\mu^3\sigma_s^2 \tag{2.1a}$$

$$b(x) = \lambda - \min(\lceil x \rceil, m)\mu \tag{2.1b}$$

are used (for example, Kimura [20], Choi and Shin [6]), where $\lceil x \rceil$ is the smallest integer not smaller than x. Let $f(x,t|x_0)$ be the probability density function of $X(t)$ given $X(0) = x_0$, i.e.

$$f(x,t|x_0)dx = P(x < X(t) \le x + dx | X(0) = x_0).$$

Throughout this paper we assume that the initial value $X(0) = x_0$ is a nonnegative integer. Then $f(x,t|x_0)$ satisfies the following Fokker-Planck equation (Feller [10], Gelenbe [12])

$$\begin{aligned}\frac{\partial f}{\partial t} =& \frac{1}{2}\frac{\partial^2}{\partial x^2}\{a(x)f(x,t|x_0)\} - \frac{\partial}{\partial x}\{b(x)f(x,t|x_0)\} \\ &+ \lambda P(t)\delta(x-1) + \sum_{i=1}^{n} \mu_i(1-c_i)Q_i(t)\delta(x-N+1),\end{aligned} \tag{2.2}$$

where $\delta(\cdot)$ is the Dirac's delta function and $P(t)$ are the probabilities that the process $X(t)$ is at the lower boundary and $Q_i(t)$ are probabilities that the process $X(t)$ is in the ith stage on the upper boundary at time t. The behaviors on the lower and upper boundaries are described by the following first order differential equations

$$\frac{dP(t)}{dt} = -\lambda P(t) + \lim_{x \downarrow 0} C_{x,t}f, \tag{2.3}$$

$$\frac{dQ_i(t)}{dt} = \begin{cases} -\mu_i Q_i(t) - \lim_{x \uparrow N} C_{x,t}f & \text{if } i = 1 \\ -\mu_i Q_i(t) + \mu_{i-1}c_{i-1}Q_{i-1}(t) & \text{if } 1 < i \le n, \end{cases} \tag{2.4}$$

where $C_{x,t}f = \frac{1}{2}\frac{\partial}{\partial x}\{a(x)f(x,t|x_0)\} - b(x)f(x,t|x_0)$.

Since the boundaries at $x = 0$ and $x = N$ behave as the absorbing boundaries during their holding times, it is natural to assume (Cox and Miller [5])

$$\lim_{x \downarrow 0} f(x,t|x_0) = 0, \tag{2.5}$$

$$\lim_{x \uparrow N} f(x,t|x_0) = 0. \tag{2.6}$$

The initial conditions for $P(t), Q_i(t)$ and $f(x,t|x_0)$ are given by

$$f(x,0|x_0) = \delta(x - x_0), \tag{2.7}$$

$$P(0) = \begin{cases} 1 & \text{if } x_0 = 0 \\ 0 & \text{if } x_0 > 0, \end{cases} \tag{2.8}$$

$$Q_i(0) = \begin{cases} 1 & \text{if } x_0 = N \text{ and } i = 1 \\ 0 & \text{if } x_0 < N \text{ or } i > 1 \end{cases} \tag{2.9}$$

Now we solve the equation (2.2) with conditions (2.3) - (2.9). For the convenience of solving partial differential equation, let $a_k = a(k)$, $b_k = b(k)$, $k = 1, 2, \cdots, m$ and $f_k(x,t|x_0)$ be the restriction of $f(x,t|x_0)$ to $k-1 < x \le k, t \ge 0, k = 1, 2, \cdots, m-1$ and $f_m(x,t|x_0)$ the restriction of $f(x,t|x_0)$ to $m - 1 < x \le N, t \ge 0$. For the notational convience we set $g_k(t|x_0) = f(k,t|x_0), k = 1, 2, \cdots, m-1$. Then from (2.5), it is easy to see that $g_0(t|x_0) = 0$.

THEOREM 1. *The Laplace transform $f^*(x,s|x_0)$ of the probability density function $f(x,t|x_0)$ of $X(t)$ with respect to t variable is given by for $k-1 < x \le k, k = 1, 2, \cdots, m-1$,*

$$\begin{aligned} f_k^*(x,s|x_0) = &\exp\left(\frac{b_k}{a_k}(x-k)\right)\frac{\sinh A_k(x-k+1)}{\sinh A_k} g_k^*(s|x_0) \\ &- \exp\left(\frac{b_k}{a_k}(x-k+1)\right)\frac{\sinh A_k(x-k)}{\sinh A_k} g_{k-1}^*(s|x_0) \end{aligned} \tag{2.10}$$

and for $m-1 < x \le N$, $t \ge 0$

(2.11)

$$\begin{aligned} f_m^*(x,s|x_0) = &\, e^{\frac{b_m}{a_m}(x-m+1)} \frac{\sinh A_m(N-x)}{\sinh A_m(N-m+1)} g_{m-1}^*(s|x_0) \\ &+ \frac{2}{a_m A_m} e^{\frac{b_m}{a_m}(x-m+1)} \left(C_{m,N} g_{m-1}^*(s|x_0) - B_{m-1} e^{2\frac{b_{m-1}}{a_{m-1}}} g_{m-2}^*(s|x_0)\right) \\ &\times \Big(\sinh A_m(x-m+1) - \frac{\sinh A_m(N-m+1)}{\sinh A_m} \\ &\qquad \sinh A_m(x-N+1) 1(x \ge N-1)\Big) \\ &+ \frac{2}{a_m A_m} e^{\frac{b_m}{a_m}(x-x_0)} \Big(\frac{\sinh A_m(N-x_0)}{\sinh A_m} \sinh A_m(x-N+1) \\ &\quad 1(x \ge N-1) - \sinh A_m(x-x_0) 1(x \ge x_0)\Big) 1(m-1 \le x_0 < N), \end{aligned}$$

where $1(D)$ *is the indicator of* D *and*

$$A_k = \sqrt{2a_k s + b_k^2}/a_k,\ k = 1,2,\cdots,m$$

$$B_k = \frac{a_k A_k}{2} e^{-\frac{b_k}{a_k}} \frac{1}{\sinh A_k},\ k = 1,2,\cdots,m-1$$

$$C_k = -\frac{b_{k-1}}{2} + \frac{a_{k-1}A_{k-1}}{2}\frac{\cosh A_{k-1}}{\sinh A_{k-1}} + \frac{b_k}{2} + \frac{a_k A_k}{2}\frac{\cosh A_k}{\sinh A_k},$$

$$k = 2,3,\cdots,m$$

$$B_{m,N} = \frac{a_m A_m}{2} e^{-\frac{b_m}{a_m}(N-m+1)} \frac{1}{\sinh A_m(N-m+1)}$$

$$C_{m,N} = -\frac{b_{m-1}}{2} + \frac{a_{m-1}A_{m-1}}{2}\frac{\cosh A_{m-1}}{\sinh A_{m-1}} + \frac{b_m}{2} + \frac{a_m A_m}{2} \frac{\cosh A_m(N-m+1)}{\sinh A_m(N-m+1)}.$$

The Laplace transforms $P^*(s)$ *and* $g_k^*(s|x_0)$ *of* $P(t)$ *and* $g_k(t|x_0)$ *are related as follows:*

(2.12) $$(\lambda + s)P^*(s) - B_1 g_1^*(s|x_0) = 1(x_0 = 0)$$

(2.13)
$$-\lambda P^*(s) + C_2 g_1^*(s|x_0) - B_2 g_2^*(s|x_0) = 1(x_0 = 1)$$

$$-B_{k-1}e^{2\frac{b_{k-1}}{a_{k-1}}} g_{k-2}^*(s|x_0) + C_k g_{k-1}^*(s|x_0)$$

(2.14) $$-B_k g_k^*(s|x_0) = 1(x_0 = k-1), \quad k = 3,4,\cdots,m-1$$

(2.15)
$$-g_{m-2}^*(s|x_0)B_{m-1}e^{2\frac{b_{m-1}}{a_{m-1}}}(\sinh A_m(N-m+1)$$
$$-e^{\frac{b_m}{a_m}}\sinh A_m(N-m)h^*(s)) + g_{m-1}^*(s|x_0)$$
$$\Big(C_{m,N}(\sinh A_m(N-m+1) - h^*(s)e^{\frac{b_m}{a_m}}\sinh A_m(N-m))$$
$$-B_{m,N}h^*(s)\sinh A_m e^{\frac{b_m}{a_m}(N-m+2)}\Big)$$
$$=\Big(h^*(s)e^{\frac{b_m}{a_m}(m-x_0)}\sinh A_m \frac{\sinh A_m(x_0 - m + 1)}{\sinh A_m(N-m+1)}$$

$$+ e^{\frac{b_m}{a_m}(m-1-x_0)} \sinh A_m(N-x_0) \times \Big(1 - h^*(s) e^{\frac{b_m}{a_m}} \frac{\sinh A_m(N-m)}{\sinh A_m(N-m+1)}\Big)\Big) 1(m-1 \le x_0 \le N).$$

The proof of theorem 1 is given in appendix.

REMARKS. 1. By letting $N \to \infty$, Theorem 1 is reduced to proposition 1 (Choi and Shin [6]).

2. When $N = m$, (2.11) and (2.15) are simplified as follows

$$f_m^*(x,s|x_0) = e^{\frac{b_m}{a_m}(x-m+1)} \frac{\sinh A_m(m-x)}{\sinh A_m} g_{m-1}^*(s|x_0),$$

and

$$\begin{aligned} &- g_{m-2}^*(s|x_0) B_{m-1} e^{2\frac{b_{m-1}}{a_{m-1}}} + g_{m-1}^*(s|x_0)\big(C_{m,m} - B_{m,m} h^*(s) e^{2\frac{b_m}{a_m}}\big) \\ =& 1(x_0 = m-1) + h^*(s) 1(x_0 = m) \end{aligned}$$

From (2.12) - (2.15) we see that $P^*(s)$, $g_1^*(s|x_0), \cdots, g_{m-1}^*(s|x_0)$ can be expressed in terms of $h^*(s)$. Thus $f_k^*(x,s|x_0)$, $k = 1, 2, \cdots, m$ are expressed in terms of $h^*(s)$.

Since the set of Cox distributions is dense in the set of distributions on $(0, \infty)$, from the continuity theorem for Laplace transform, the formulas of $f_k^*(x,s|x_0)$ $(k = 1, 2, \cdots, m)$ expressed in terms of $h^*(s)$ remain valid not only for Cox distribution of holding time but also for general distribution of holding time at the upper boundary.

The state in which the capacity is fully occupied is called the loss state, and probability that the system is in the loss state is called the loss probability. Let $P_N(t)$ be the probability that the process $X(t)$ is on the upper boundary $x = N$ at time t. We use the probability $P_N(t)$ to approximate the loss probability of original queueing system at time t. The probability $P_N(t)$ is obtained from the conservation of probability

$$P(t) + \int_0^N f(x,t|x_0)\,dx + P_N(t) = 1.$$

Taking Laplace transform and then applying theorem 1, we have

$$\begin{aligned} P_N^*(s) =& \frac{1}{s} - \Big(P^*(s) + \int_0^N f^*(x,s|x_0)\,dx\Big) \\ =& \frac{1}{s} - \Big(\sum_{k=1}^{m-1} \int_{k-1}^{k} f_k^*(x,s|x_0)dx + \int_{m-1}^{N} f_m^*(x,s|x_0)dx + P^*(s)\Big) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{s} - \frac{1}{s}\sum_{k=1}^{m-1} \frac{a_k A_k}{2} \Big[g_k^*(s|x_0) \Big(\frac{\cosh A_k}{\sinh A_k} - \frac{b_k}{a_k A_k} - \frac{e^{-\frac{b_k}{a_k}}}{\sinh A_k} \Big) \\
&\qquad + g_{k-1}^*(s|x_0) \Big(\frac{\cosh A_k}{\sinh A_k} + \frac{b_k}{a_k A_k} - \frac{e^{\frac{b_k}{a_k}}}{\sinh A_k} \Big) \Big] \\
&- \frac{a_m A_m}{2s} g_{m-1}^*(s|x_0) \Big(\frac{\cosh A_m(N-m+1)}{\sinh A_m(N-m+1)} + \frac{b_m}{a_m A_m} \\
&\qquad - \frac{e^{\frac{b_m}{a_m}(N-m+1)}}{\sinh A_m(N-m+1)} \Big) \\
&- \frac{1}{s} \Big(C_{m,N} g_{m-1}^*(s|x_0) - B_{m-1} e^{2\frac{b_{m-1}}{a_{m-1}}} g_{m-2}^*(s|x_0) \Big) \\
&\qquad \times \Big(e^{\frac{b_m}{a_m}(N-m)} - e^{\frac{b_m}{a_m}(N-m+1)} \frac{\sinh A_m(N-m)}{\sinh A_m} - 1 \Big) \\
&- \frac{1}{s} \Big(e^{\frac{b_m}{a_m}(N-x_0)} \frac{\sinh A_m(N-1-x_0)}{\sinh A_m} - e^{\frac{b_m}{a_m}(N-1-x_0)} + 1 \Big) \\
&\qquad \times 1(m-1 \le x_0 < N) - P^*(s).
\end{aligned}
$$

Next we derive the stationary solution. In the system with finite capacity the stationary distribution always exists. Let $f(x) = \lim_{t\to\infty} f(x,t|x_0)$, $P = \lim_{t\to\infty} P(t)$ and $P_N = \lim_{t\to\infty} P_N(t)$. To obtain the limiting probability from the Laplace transform solution, we use the final value theorem for Laplace transform: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s f^*(s)$.

THEOREM 2. *The stationary probability density function $f(x)$ of $X(t)$ is given by;*
(2.17)

$$
f(x) = \begin{cases}
\frac{e^{\gamma_1 x}-1}{\gamma_1}\frac{2\lambda}{a_1} P & \text{if } 0 < x \le 1 \\
q_k e^{\gamma_k (x-k)} P \quad \text{if } k-1 < x < k, & k = 2,3,\cdots,m-1, \\
q_m e^{\gamma_m (x-m)} P & \text{if } m-1 < x < N-1, \\
q_m e^{\gamma_m (N-m)} \frac{1-e^{\gamma_m (x-N)}}{e^{\gamma_m}-1} P & \text{if } N-1 < x < N,
\end{cases}
$$

where $\gamma_k = \frac{2b_k}{a_k}$, $k = 1,2,\cdots,m$ and $q_1 = \frac{e^{\gamma_1}-1}{\gamma_1}\frac{2\lambda}{a_1}$, $q_k = q_1 \exp(\sum_{j=2}^{k} \gamma_j)$, $k = 2,3,\cdots,m$.

The probability P that the process $X(t)$ is in the lower boundary is *given by;*

$$P = \Big[1 + \frac{e^{\gamma_1} - 1 - \gamma_1}{\gamma_1^2}\frac{2\lambda}{a_1} + \sum_{k=2}^{m-1} q_{k-1}\frac{e^{\gamma_k} - 1}{\gamma_k} + q_{m-1}\frac{e^{\gamma_m(N-m)} - 1}{\gamma_m} + q_m e^{\gamma_m(N-m)}\Big(\frac{\gamma_m}{e^{\gamma_m} - 1}\frac{a_m}{2}\frac{1}{\Lambda} + \frac{e^{-\gamma_m} - 1 + \gamma_m}{(e^{\gamma_m} - 1)\gamma_m}\Big)\Big]^{-1} \tag{2.18}$$

The probability P_N is given by

$$P_N = \begin{cases} \frac{1}{\Lambda}\frac{b_m}{e^{\gamma_m}-1}e^{\gamma_m(N-m+1)}q_{m-1}\lambda P & \text{if } b_m \neq 0 \\ \frac{1}{\Lambda}\frac{a_m}{2}q_{m-1}\lambda P & \text{if } b_m = 0, \end{cases} \tag{2.19}$$

where $\frac{1}{\Lambda}$ is the mean holding time at the boundary $x = N$.

For the proof of theorem 2, see the appendix. Note that the results (2.17) - (2.19) can be obtained by letting $t \to \infty$ in (2.2) -(2.7) and solving the equation (2.2) with the conditions (2.3) - (2.7).

3. Numerical examples and conclusion

In order to examine the accuracy of approximations obtained in section 2, we compare numerically diffusion approximation with simulation. Since we have approximated a discrete state process with a continuous state process, we discretize the probability density function $f(x,t|x_0)$ derived in section 2. The discretization of probability density function $f(x,t|x_0)$ can be done in serval different ways (see Gelenbe and Pujolle [14]). Modifying the method used in Duda [9] and Choi and Shin [6, 7] so as $\sum_{n=0}^{N} p_i(t) = 1$, we adopt the following;

$$\begin{aligned} p_0(t) &:= P(t)/Total \\ p_n(t) &:= f(n,t)/Total,\ n = 1, 2, \cdots, N-1, \\ p_N(t) &:= P_N(t)/Total \end{aligned} \tag{3.1}$$

where $Total := P(t) + P_N(t) + \sum_{i=1}^{N-1} f(n,t)$. From the argument on formulating the diffusion process in section 2, we should determine the distribution of holding times at the boundaries. For $M/G/m/N$ system, the idle period is exponential with parameter λ. The distribution of the time period that the system is full in the $M/G/m/N$ system, we have the following possible two choices

$$h_{N,1}(t) = m(1 - S(t))^{m-1} s(t) \tag{3.2}$$

(3.3)

$$h_{N,2}(t) = \begin{cases} m(1 - S_e(t))^{m-1} s_e(t) & \text{if } m < N \\ (1 - S_e(t))^{m-2} & \\ \times\big((m-1)s_e(t)(1 - S(t)) + (1 - S_e(t))s(t)\big) & \text{if } m = N, \end{cases}$$

where $S_e(t) = \mu \int_0^t (1 - S(\tau))\, d\tau$ is the distribution of remaining service time and $s_e(t) = \mu(1 - S(t))$ is its p.d.f. (3.3) is obtained from the following assumption (Nozaki and Ross [23]).

Given that a customer arrives to find i busy servers ($i > 0$), then at the time that he enters service, the remaining service time of other $i - \delta(i, m)$ customers being served has a joint distribution that is approximately of independent random variables each having distribution $S_e(t)$, where $\delta(i, j)$ is the Kronecker delta.

It was reported from many simulations that for the single server system with finite capacity , the approximate results using the service time $S(t)$ is more accurate than the results using $S_e(t)$ (Kimura [17]). This is a reason why we choose (3.2) as holding time at upper boundary.

For $M/M/m/N$ system, the holding time at the upper boundary $x = N$ is exponential with parameter $m\mu$ from the memoryless property of exponential distribution. Note that when $S(t) = 1 - e^{-\mu t}$, i.e. the service time is exponential, $h_{N,1}(t) = h_{N,2}(t) = m\mu e^{-m\mu t}$, which coincides with the exact results. The numerical inversion of Laplace transform is derived by using the Algorithm 368 in ACM [23]. The simulation results (denoted by "sim" in tables) are obtained by 30,000 times repetitions. If there are i customers in the system at time t during the jth repetition of simulation, let $I_{ij}(t) = 1$ otherwise $I_{ij}(t) = 0$. Let

$$\bar{P}_{ij}(t) = \frac{1}{2000} \sum_{k=(j-1)2000+1}^{2000j} I_{ik}(t), \ j = 1, 2, \cdots, 15$$

$$\bar{P}_i(t) = \frac{1}{30000} \sum_{j=1}^{30000} I_{ij}(t)$$

$$\hat{\sigma}^2 = \sum_{j=1}^{15} (\bar{P}_{ij}(t) - \bar{P}_i(t))^2/(15-1)$$

Then the 95% confidence interval (denoted "c.i." in tables) of $P_i(t)$ is obtained by

$$\bar{P}_i(t) \pm t(14, 0.025)\frac{\hat{\sigma}}{\sqrt{15}},$$

where $t(14, 0.025)$ is the 97.5 upper percentage point of t-distribution with 14 degrees of freedom.

Tables 1 - 3 represent the comparisons of the diffusion approximation with the simulation for $M/M/3/N$ queueing system for various traffic intensities $\rho = 0.7, 0.95, 1.2$. In tables $P_i(t)$ denotes the probability that the number of customers in the system at time t is i and $P_B(t)$ is the probability that all servers are busy at time t. In tables 4 and 5, we present the numerical comparison for $M/E_2/3/N$ system with mean service time $\frac{1}{\mu} = 1.0$. The numerical results for $M/H_2/3/N$ system are presented in Tables 6 and 7. In Tables 7 and 8 we use the service time density function $s(t) = p_1\mu_1 e^{-\mu_1 t} + p_2\mu_2 e^{-\mu_2 t}$, where $p_1 = 0.5(1+\sqrt{0.2}), p_2 = 1 - p_1$ and $\mu_1 = 2p_1, \mu_2 = 2p_2$. In Tables 4 - 7, "diff1" denotes the diffusion approximation results when the holding time distribution is $h_{N,1}(t)$ and "diff2" is the corresponding to the holding time distribution $h_{N,2}(t)$ at the upper boundary. The results represented in tables 5 - 7 show that the "diff1" is more accurate than that of "diff2". From the numerical results it is concluded that the diffusion approximation gives very accurate results for exponential service time. Our approximation for nonexponential service system is slightly less accurate than for exponential service system. We think that approximation of holding time at upper boundary causes comparatively low accuracy. It is noted by comparing the transient probability $p_k(t)$ with the stationary probablity $p_k(\infty)$ in tables 1 - 4 that the smaller the system capacity N is, the shorter it takes for the process to reach the stationary state.

The method and results for transient diffusion approximation for $M/G/m/N$ system can be applied to analyze the transient behavior of two-stage cyclic queueing system.

Appendix

Proof of Theorem 1. Theorem 1 can be proved by combining the arguments used for derivation of the transient solution in Choi and Shin [6, 7]. However for the completion of the paper we sketch the proof of theorem 1. Since there exists a continuous solution of the equation (2.2) even if the functions $a(x)$ and $b(x)$ are piecewise constants, we impose the condition

$$\lim_{x \downarrow k-1} f_k(x,t|x_0) = f_{k-1}(k-1,t|x_0), k = 2,3,\cdots,m.$$

Then the problem of solving the differential equation (2.2) is reduced to the following initial boundary value problems, for $k-1 < x < k, t > 0, k = 1,2,\cdots,m-1$,

$$\frac{\partial f_k}{\partial t} = \frac{1}{2} a_k \frac{\partial^2 f_k}{\partial x^2} - b_k \frac{\partial f_k}{\partial x} \tag{A.1}$$

$$f_k(k-1,t|x_0) = g_{k-1}(t|x_0) \tag{A.2a}$$

$$f_k(k,t|x_0) = g_k(t|x_0) \tag{A.2b}$$

$$f_k(x,0|x_0) = \delta(x-x_0) \tag{A.2c}$$

and for $m-1 < x < N, t > 0$

$$\frac{\partial f_m}{\partial t} = \frac{1}{2} a_m \frac{\partial^2 f_m}{\partial x^2} - b_m \frac{\partial f_m}{\partial x} + \sum_{i=1}^{\eta} \mu_i (1-c_i) Q_i(t) \delta(x-N+1) \tag{A.3}$$

$$f_m(m-1,t|x_0) = g_{m-1}(t|x_0) \tag{A.4a}$$

$$f_m(N,t|x_0) = 0 \tag{A.4b}$$

$$f_m(x,0|x_0) = \delta(x-x_0). \tag{A.4c}$$

The Laplace transform of solution of (A.1) with the condition (A.2) with respect to t variable is (2.10) and that of (A.3) with the condition (A.4)

(A.5)

$$f_m^*(x,s|x_0) = e^{\frac{b_m}{a_m}(x-m+1)} \frac{\sinh A_m(N-x)}{\sinh A_m(N-m+1)} g_{m-1}^*(s|x_0)$$

$$+ \frac{2}{a_m A_m} e^{\frac{b_m}{a_m}(x-N+1)} \Big(\frac{\sinh A_m}{\sinh A_m(N-m+1)} \sinh A_m(x-m+1)$$

$$- \sinh A_m(x - N + 1)1(x \geq N - 1)\Big)\Big(\sum_{i=1}^{n} \mu_i(1 - c_i)Q_i^*(s)\Big)$$
$$+ \frac{2}{a_m A_m} e^{\frac{b_m}{a_m}(x - x_0)}\Big(\frac{\sinh A_m(N - x_0)}{\sinh A_m(N - m + 1)} \sinh A_m(x - N + 1)$$
$$- \sinh A_m(x - x_0)1(x \geq x_0)\Big)1(m - 1 < x_0 < N).$$

For the derivation of (2.10) and (A.5), see the appendix of Choi and Shin [6]. Taking the Laplace transform with respective to t-variable of the equation (2.3) and (2.4), we have the following relations

$$\text{(A.6)} \qquad [C_{x,s}f_1^*]_{x\downarrow 0} = (\lambda + s)P^*(s) - P(0)$$

and

$$\text{(A.7)} \qquad [C_{x,s}f_m^*]_{x\uparrow N} = Q_1(0) - (\mu_1 + s)Q_1^*(s)$$

$$\text{(A.8)} \qquad Q_i^*(s) = \frac{d_i}{\mu_i} e_i^*(s)(\mu_1 + s)Q_1^*(s), \ 1 < i \leq n$$

where

$$C_{x,s}f^* = \frac{1}{2}\frac{\partial}{\partial x}\{a(x)f^*(x, t|x_0)\} - b(x)f^*(x, s|x_0).$$

Multipling both sides of (A.8) by $\mu_i(1 - c_i)$ and then summing over i, we have

$$\text{(A.9)} \qquad \sum_{i=1}^{n} \mu_i(1 - c_i)Q_i^*(s) = \sum_{i=1}^{n} d_i(1 - c_i)e_i^*(s)(\mu_1 + s)Q_1^*(s)$$
$$= h^*(s)(\mu_1 + s)Q_1^*(s).$$

To determine $g_k^*(s|x_0)$'s and $P^*(s)$ in terms of known parameters, we take the Laplace transform of equation (2.2) with respect to t-variable, and then integrate with respect to x variable. Then we have

(A.10)

$$C_{x,s}f^* = [C_{x,s}f_1^*]_{x\downarrow 0} + s\int_0^x f^*(y, s|x_0)dy - 1(x \geq x_0)$$
$$- \lambda P^*(s)1(x \geq 1) - \sum_{i=1}^{n} \mu_i(1 - c_i)Q_i^*(s)1(x \geq N - 1).$$

After simple calculation we have from (A.9) and (A.10) that

(A.11) $$[C_{x,s}f_2^*]_{x\downarrow 1} = [C_{x,s}f_1^*]_{x\uparrow 1} - \lambda P^*(s) - 1(x_0 = 1)$$

(A.12) $$[C_{x,s}f_k^*]_{x\downarrow k-1} = [C_{x,s}f_{k-1}^*]_{x\uparrow k-1} - 1(x_0 = k-1), \quad k = 3,4,\cdots,m-1$$

(A.13) $$\begin{aligned}[C_{x,s}f_m^*]_{x\downarrow m-1} = [C_{x,s}f_{m-1}^*]_{x\uparrow m-1} &- 1(x_0 = m-1) \\ &- h^*(s)(\mu_1 + s)Q_1^*(s)1(m = N).\end{aligned}$$

Calculating $C_{x,s}f_1^*$ from (2.10) and then substituting it into (A.6), we have (2.12). Similarly we obtain (2.13) and (2.14) by calculating $C_{x,s}f_k^*$ and substituting it into (A.11) and (A.12), respectively. Doing the same thing for (A.5) and (A.7), (A.5) and (A.13) yields the followings

(A.14)
$$\begin{aligned}&h^*(s)(\mu_1 + s)Q_1^*(s)\frac{e^{-\frac{b_m}{a_m}(N-m)}\sinh A_m}{\sinh A_m(N-m+1)} \\ &= C_{m,N}g_{m-1}^*(s|x_0) - B_{m-1}e^{2\frac{b_{m-1}}{a_{m-1}}}g_{m-2}^*(s|x_0) \\ &\quad - e^{\frac{b_m}{a_m}(m-1-x_0)}\frac{\sinh A_m(N-x_0)}{\sinh A_m(N-m+1)}1(m-1 \le x_0 < N)\end{aligned}$$

(A.15)
$$\begin{aligned}&(\mu_1 + s)Q_1^*(s)\left(1 - h^*(s)e^{\frac{b_m}{a_m}}\frac{\sinh A_m(N-m)}{\sinh A_m(N-m+1)}\right) \\ &= g_{m-1}^*(s|x_0)B_{m,N}e^{2\frac{b_m}{a_m}(N-m+1)} \\ &\quad + e^{\frac{b_m}{a_m}(N-x_0)}\frac{\sinh A_m(x_0-m+1)}{\sinh A_m(N-m+1)}1(m-1 < x_0 \le N).\end{aligned}$$

By eliminating $\sum_{i=1}^n(\mu_i + s)Q_i^*(s)$ in (A.5) using (A.9) and (A.14), we get (2.11) and from (A.14) and (A.15) we have (2.15).

Proof of Theorem 2. Th proof of theorem 2 is similar to the poposition 2 in Choi and Shin [6] except the calculation of the probabilities P and P_N and we simply sketch the proof. Simple calculation yields

$$\lim_{s\to 0} A_k = \frac{|b_k|}{a_k}, k = 1, 2, \cdots, m,$$

$$\lim_{s\to 0} B_k = \begin{cases} \frac{b_k}{e^{2\frac{b_k}{a_k}}-1} & \text{if } b_k \neq 0, k = 1, 2, \cdots, m-1, \\ \frac{a_k}{2} & \text{if } b_k = 0, \end{cases}$$

$$\lim_{s\to 0} B_{m,N} = \begin{cases} \frac{b_m}{e^{2\frac{b_m}{a_m}}-1} & \text{if } b_m \neq 0, \\ \frac{a_m}{2(N-m+1)} & \text{if } b_m = 0, \end{cases}$$

$$\lim_{s\to 0} C_k = B_{k-1}(0) + B_{m,N}(0) e^{2\frac{b_m}{a_m}(N-m+1)}.$$

Then from (2.12) we have

$$g_1 = \lim_{s\to 0} s g_1^*(s|x_0) = \frac{\lambda}{b_1}(e^{2\frac{b_1}{a_1}} - 1)P. \tag{A.16}$$

Similarly we have from (2.13) and (A.16)

$$g_2 = \lim_{s\to 0} s g_2^*(s|x_0) = e^{2\frac{b_2}{a_2}} g_1. \tag{A.17}$$

We have from (2.14) that for $k = 3, 4, \cdots, m-1$

$$g_k = \frac{e^{2\frac{b_k}{a_k}} - 1}{b_k} \frac{b_{k-1}}{e^{2\frac{b_{k-1}}{a_{k-1}}} - 1}(g_{k-1} - e^{2\frac{b_{k-1}}{a_{k-1}}} g_{k-2}) + e^{2\frac{b_k}{a_k}} g_{k-1}. \tag{A.17}$$

Since $g_2 = e^{2\frac{b_2}{a_2}} g_1$, (A.17) with $k = 3$ gives $g_3 = e^{2\frac{b_3}{a_3}} g_2$. By induction we obtain

$$g_k = e^{2\frac{b_k}{a_k}} g_{k-1} = g_1 \prod_{j=2}^{k} e^{2\frac{b_k}{a_k}}, \quad k = 2, 3, \cdots, m-1, \tag{A.18}$$

Using (A.16) and (A.18), we obtain the first two cases of (2.17). Simple but tedious calculation of $\lim_{s\to 0} f_m^*(x, s|x_0)$ from (2.11) yields the least two cases of (2.17). For the calculation of the probabilities P and P_N first we consider the case of Cox distribution of holding time at the upper boundary. Multiplying both sides of (A.15) by s and letting $s \to 0$, we have

$$\mu_1 Q_1 = \begin{cases} g_{m-1} \dfrac{b_m e^{2\frac{b_m}{a_m}(N-m+1)}}{e^{2\frac{b_m}{a_m}}-1} & \text{if } b_m \neq 0 \\ \frac{a_m}{2} g_{m-1} & \text{if } b_m = 0. \end{cases}$$

Similarly we have from (A.8) that

$$Q_i = \lim_{s\to 0} sQ_i^*(s) = \frac{d_i}{\mu_i}\mu_1 Q_1, i = 1, 2, \cdots, n.$$

The stationary probability that the process $X(t)$ is at $x = N$ is

$$\text{(A.19)} \quad \begin{aligned} P_N &= \sum_{i=1}^{n} Q_i = \sum_{i=1}^{n} \frac{d_i}{\mu_i}\mu_1 Q_1 = \frac{\mu_1 Q_1}{\Lambda} \\ &= \begin{cases} \frac{1}{\Lambda}\dfrac{b_m}{e^{2\frac{b_m}{a_m}}-1} e^{2\frac{b_m}{a_m}(N-m+1)} q_{m-1}\lambda P & \text{if } b_m \neq 0 \\ \frac{1}{\Lambda}\frac{a_m}{2} q_{m-1}\lambda P & \text{if } b_m = 0. \end{cases} \end{aligned}$$

The porbability P is determined from the conservation of probability, i.e.

$$\begin{aligned} 1 &= P + \int_0^N f(x)dx + P_N \\ &= P + \sum_{k=1}^{m-1} \int_{k-1}^{k} f_k(x)dx + \int_{m-1}^{N} f_m(x)dx + P_N \end{aligned}$$

Now we consider the case of general distribution of holding time. When we consider the formula (2.19) as a function of holding time distribution at the upper boundary, it depends only on the first moment of holding time. From the fact that if a distribuiton $F(t)$ has finite q-th moment, i.e. $\int_0^\infty x^q dF(x) < \infty$, then there exists a sequence of Cox distributions $\{F_k\}$ such that

$$\int_0^\infty x^p dF_k(x) \to \int_0^\infty x^p dF(x)$$

for $0 \leq p \leq q$ (e.g. see Asmusen [3]), the formula (2.19) holds for arbitrary distribution of holding time with finite first moment.

TABLE 1

Comparison of the diffusion results with simulation for $M/M/3/N$ Queue ($x_0 = 0$, $\mu = 1.0$, $\rho = 0.7$)

t	N / prob. / method	3: $P_0(t)$	3: $P_1(t)$	3: $P_2(t)$	3: $P_N(t)$	8: $P_0(t)$	8: $P_1(t)$	8: $P_2(t)$	8: $P_B(t)$	8: $P_N(t)$
0.3	diff	0.544	0.341	0.102	0.012	0.542	0.340	0.101	0.018	0.000
	sim	0.577	0.318	0.087	0.018	0.579	0.317	0.085	0.018	0.000
	c.i.	0.006	0.008	0.002	0.002	0.005	0.005	0.003	0.001	0.000
0.5	diff	0.422	0.364	0.169	0.045	0.419	0.361	0.165	0.053	0.000
	sim	0.438	0.364	0.150	0.048	0.437	0.363	0.147	0.052	0.000
	c.i.	0.006	0.006	0.005	0.002	0.006	0.004	0.004	0.003	0.000
0.7	diff	0.342	0.361	0.215	0.083	0.340	0.356	0.204	0.099	0.000
	sim	0.350	0.370	0.196	0.085	0.348	0.367	0.193	0.091	0.000
	c.i.	0.006	0.006	0.005	0.004	0.007	0.007	0.005	0.004	0.000
1.0	diff	0.268	0.345	0.256	0.131	0.266	0.337	0.236	0.160	0.000
	sim	0.265	0.357	0.248	0.130	0.267	0.348	0.233	0.152	0.000
	c.i.	0.005	0.005	0.006	0.003	0.004	0.005	0.004	0.005	0.000
3.0	diff	0.158	0.305	0.315	0.221	0.137	0.251	0.245	0.366	0.010
	sim	0.151	0.309	0.319	0.221	0.131	0.259	0.249	0.361	0.010
	c.i.	0.005	0.006	0.004	0.005	0.004	0.006	0.003	0.005	0.001
5.0	diff	0.153	0.303	0.318	0.226	0.117	0.226	0.232	0.425	0.019
	sim	0.146	0.306	0.320	0.228	0.113	0.230	0.236	0.421	0.020
	c.i.	0.004	0.006	0.008	0.004	0.003	0.004	0.005	0.006	0.001
7.0	diff	0.153	0.303	0.318	0.226	0.111	0.217	0.226	0.445	0.024
	sim	0.145	0.305	0.324	0.226	0.104	0.222	0.230	0.444	0.023
	c.i.	0.004	0.006	0.005	0.006	0.004	0.005	0.004	0.005	0.002
10.0	diff	0.154	0.303	0.318	0.226	0.108	0.212	0.222	0.458	0.026
	sim	0.144	0.306	0.321	0.230	0.100	0.217	0.228	0.455	0.026
	c.i.	0.005	0.005	0.006	0.006	0.003	0.005	0.005	0.007	0.002
20.0	diff	0.153	0.303	0.318	0.225	0.107	0.211	0.221	0.461	0.027
	sim	0.146	0.308	0.321	0.226	0.101	0.214	0.224	0.460	0.026
	c.i.	0.005	0.004	0.006	0.005	0.004	0.005	0.004	0.006	0.002
∞	diff	0.153	0.303	0.318	0.225	0.107	0.211	0.222	0.461	0.027
	exact	0.146	0.307	0.322	0.225	0.102	0.213	0.224	0.461	0.026

TABLE 2

Comparison of the diffusion results with simulation for $M/M/3/N$ Queue

$(x_0 = 0,\ \mu = 1.0,\ \rho = 0.95)$

t	N / prob. / method	3: $P_0(t)$	3: $P_1(t)$	3: $P_2(t)$	3: $P_N(t)$	8: $P_0(t)$	8: $P_1(t)$	8: $P_2(t)$	8: $P_B(t)$	8: $P_N(t)$
0.3	diff	0.457	0.361	0.153	0.029	0.453	0.358	0.149	0.039	0.000
	sim	0.478	0.354	0.131	0.037	0.478	0.353	0.130	0.039	0.000
	c.i.	0.007	0.006	0.004	0.003	0.006	0.006	0.004	0.002	0.000
0.5	diff	0.326	0.353	0.229	0.092	0.323	0.349	0.218	0.110	0.000
	sim	0.325	0.365	0.213	0.097	0.324	0.365	0.203	0.108	0.000
	c.i.	0.007	0.007	0.004	0.004	0.007	0.006	0.005	0.003	0.000
0.7	diff	0.247	0.328	0.271	0.154	0.245	0.321	0.249	0.185	0.000
	sim	0.235	0.347	0.262	0.160	0.236	0.342	0.241	0.181	0.000
	c.i.	0.005	0.006	0.005	0.004	0.005	0.008	0.005	0.003	0.000
1.0	diff	0.178	0.293	0.304	0.224	0.176	0.279	0.262	0.283	0.001
	sim	0.164	0.306	0.304	0.226	0.163	0.296	0.261	0.280	0.002
	c.i.	0.005	0.006	0.006	0.005	0.005	0.005	0.006	0.004	0.001
3.0	diff	0.098	0.241	0.340	0.321	0.069	0.155	0.201	0.575	0.043
	sim	0.086	0.245	0.342	0.327	0.063	0.158	0.202	0.576	0.045
	c.i.	0.004	0.005	0.005	0.005	0.002	0.004	0.005	0.006	0.002
5.0	diff	0.096	0.240	0.340	0.324	0.051	0.122	0.166	0.661	0.078
	sim	0.084	0.243	0.348	0.324	0.044	0.121	0.171	0.664	0.077
	c.i.	0.004	0.006	0.004	0.007	0.003	0.003	0.004	0.005	0.003
7.0	diff	0.096	0.240	0.340	0.323	0.045	0.109	0.152	0.694	0.094
	sim	0.085	0.241	0.343	0.332	0.039	0.110	0.153	0.699	0.095
	c.i.	0.004	0.006	0.006	0.006	0.003	0.003	0.006	0.006	0.004
10.0	diff	0.096	0.239	0.341	0.324	0.041	0.102	0.145	0.711	0.102
	sim	0.085	0.243	0.345	0.327	0.034	0.102	0.148	0.715	0.102
	c.i.	0.006	0.006	0.008	0.007	0.002	0.004	0.004	0.006	0.003
20.0	diff	0.096	0.240	0.340	0.323	0.040	0.100	0.142	0.718	0.105
	sim	0.083	0.243	0.348	0.326	0.035	0.102	0.142	0.721	0.104
	c.i.	0.002	0.007	0.007	0.007	0.002	0.004	0.003	0.005	0.004
∞	diff	0.096	0.240	0.340	0.323	0.040	0.100	0.142	0.717	0.105
	exact	0.085	0.242	0.345	0.328	0.035	0.101	0.143	0.721	0.105

TABLE 3

Comparison of the diffusion results with simulation for $M/M/3/N$ Queue ($x_0 = 0$, $\mu = 1.0$, $\rho = 1.2$)

t	N / prob. / method	3: $P_0(t)$	$P_1(t)$	$P_2(t)$	$P_N(t)$	8: $P_0(t)$	$P_1(t)$	$P_2(t)$	$P_B(t)$	$P_N(t)$
0.3	diff	0.386	0.361	0.198	0.055	0.382	0.356	0.192	0.069	0.000
	sim	0.398	0.364	0.173	0.064	0.396	0.367	0.169	0.068	0.000
	c.i.	0.007	0.006	0.005	0.003	0.007	0.007	0.005	0.003	0.000
0.5	diff	0.254	0.323	0.269	0.152	0.251	0.318	0.251	0.180	0.000
	sim	0.247	0.342	0.252	0.159	0.246	0.338	0.244	0.172	0.000
	c.i.	0.006	0.005	0.003	0.006	0.006	0.007	0.004	0.005	0.000
0.7	diff	0.180	0.282	0.301	0.236	0.178	0.272	0.265	0.285	0.000
	sim	0.167	0.301	0.296	0.235	0.167	0.296	0.263	0.274	0.001
	c.i.	0.004	0.005	0.007	0.006	0.004	0.007	0.004	0.005	0.001
1.0	diff	0.122	0.239	0.322	0.318	0.118	0.217	0.252	0.413	0.004
	sim	0.109	0.250	0.318	0.323	0.105	0.233	0.259	0.403	0.006
	c.i.	0.003	0.006	0.006	0.003	0.004	0.006	0.006	0.007	0.001
3.0	diff	0.066	0.191	0.337	0.406	0.033	0.086	0.135	0.745	0.115
	sim	0.053	0.197	0.342	0.408	0.027	0.088	0.136	0.749	0.115
	c.i.	0.002	0.003	0.005	0.004	0.002	0.003	0.005	0.006	0.003
5.0	diff	0.065	0.190	0.337	0.406	0.021	0.056	0.094	0.829	0.182
	sim	0.051	0.190	0.344	0.415	0.017	0.053	0.097	0.833	0.183
	c.i.	0.002	0.006	0.005	0.004	0.002	0.003	0.003	0.003	0.006
7.0	diff	0.066	0.191	0.338	0.406	0.016	0.046	0.080	0.857	0.206
	sim	0.053	0.189	0.342	0.416	0.013	0.044	0.081	0.861	0.206
	c.i.	0.002	0.003	0.006	0.006	0.002	0.002	0.003	0.005	0.005
10.0	diff	0.066	0.190	0.338	0.406	0.015	0.042	0.075	0.868	0.218
	sim	0.053	0.193	0.343	0.411	0.012	0.042	0.075	0.871	0.219
	c.i.	0.004	0.006	0.006	0.008	0.001	0.002	0.004	0.005	0.004
20.0	diff	0.065	0.190	0.338	0.406	0.014	0.041	0.073	0.871	0.218
	sim	0.054	0.189	0.340	0.417	0.012	0.040	0.071	0.878	0.218
	c.i.	0.003	0.005	0.005	0.004	0.001	0.002	0.003	0.004	0.005
∞	diff	0.066	0.191	0.338	0.406	0.014	0.041	0.073	0.871	0.219
	exact	0.053	0.191	0.344	0.412	0.011	0.041	0.073	0.874	0.219

TABLE 4

Comparison of the diffusion results with simulation for $M/E_2/3/N$ Queue
$(x_0 = 0,\ \mu = 1.0,\ \rho = 0.95)$

	N	3				8				
t	prob. method	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_N(t)$	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_B(t)$	$P_N(t)$
0.3	diff1	0.442	0.394	0.147	0.017	0.438	0.390	0.146	0.025	0.000
	diff2	0.442	0.393	0.149	0.016	0.438	0.390	0.146	0.025	0.000
	sim	0.443	0.360	0.149	0.049	0.441	0.359	0.150	0.050	0.000
	c.i.	0.006	0.006	0.004	0.002	0.007	0.006	0.004	0.002	0.000
0.5	diff1	0.314	0.390	0.228	0.068	0.307	0.382	0.229	0.083	0.000
	diff2	0.311	0.388	0.239	0.062	0.307	0.382	0.229	0.083	0.000
	sim	0.277	0.356	0.236	0.131	0.275	0.356	0.230	0.139	0.000
	c.i.	0.007	0.006	0.004	0.004	0.007	0.006	0.006	0.005	0.000
0.7	diff1	0.236	0.361	0.275	0.128	0.229	0.351	0.271	0.150	0.000
	diff2	0.233	0.360	0.294	0.114	0.229	0.351	0.271	0.150	0.000
	sim	0.191	0.318	0.283	0.208	0.189	0.316	0.261	0.235	0.001
	c.i.	0.004	0.006	0.005	0.006	0.004	0.006	0.006	0.003	0.000
1.0	diff1	0.162	0.305	0.303	0.230	0.161	0.303	0.291	0.243	0.000
	diff2	0.164	0.320	0.340	0.176	0.161	0.304	0.292	0.243	0.000
	sim	0.127	0.270	0.316	0.287	0.124	0.258	0.268	0.351	0.002
	c.i.	0.004	0.005	0.005	0.005	0.004	0.005	0.007	0.007	0.001
3.0	diff1	0.074	0.226	0.348	0.352	0.059	0.166	0.232	0.542	0.029
	diff2	0.083	0.253	0.391	0.273	0.059	0.166	0.232	0.542	0.020
	sim	0.085	0.243	0.346	0.326	0.050	0.132	0.186	0.632	0.040
	c.i.	0.002	0.006	0.006	0.006	0.003	0.005	0.006	0.006	0.002
5.0	diff1	0.072	0.226	0.352	0.350	0.044	0.129	0.191	0.637	0.066
	diff2	0.081	0.252	0.392	0.275	0.044	0.130	0.192	0.634	0.044
	sim	0.082	0.244	0.347	0.327	0.037	0.104	0.153	0.706	0.065
	c.i.	0.004	0.005	0.010	0.008	0.002	0.003	0.006	0.005	0.003
10.0	diff1	0.073	0.228	0.356	0.342	0.033	0.102	0.157	0.708	0.100
	diff2	0.081	0.252	0.393	0.275	0.034	0.105	0.162	0.699	0.062
	sim	0.085	0.242	0.349	0.325	0.029	0.088	0.134	0.749	0.084
	c.i.	0.004	0.005	0.005	0.005	0.002	0.003	0.006	0.007	0.004
20.0	diff1	0.072	0.225	0.350	0.353	0.031	0.097	0.151	0.720	0.111
	diff2	0.082	0.253	0.392	0.274	0.032	0.101	0.158	0.708	0.068
	sim	0.085	0.241	0.346	0.328	0.031	0.089	0.131	0.749	0.089
	c.i.	0.004	0.005	0.005	0.006	0.002	0.002	0.005	0.005	0.004

TABLE 5

Comparison of the diffusion results with simulation for $M/E_2/3/N$ Queue
($x_0 = 0,\ \mu = 1.0,\ \rho = 1.2$)

	N	3				8				
t	prob. method	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_N(t)$	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_B(t)$	$P_N(t)$
0.3	diff1	0.345	0.387	0.196	0.043	0.370	0.382	0.196	0.052	0.000
	diff2	0.374	0.386	0.201	0.039	0.370	0.382	0.196	0.052	0.000
	sim	0.359	0.366	0.191	0.084	0.359	0.365	0.190	0.086	0.000
	c.i.	0.006	0.006	0.005	0.004	0.007	0.006	0.005	0.004	0.000
0.5	diff1	0.243	0.344	0.265	0.148	0.238	0.340	0.270	0.152	0.000
	diff2	0.242	0.346	0.287	0.125	0.238	0.340	0.270	0.152	0.000
	sim	0.203	0.318	0.271	0.209	0.202	0.319	0.258	0.222	0.000
	c.i.	0.006	0.006	0.005	0.004	0.005	0.007	0.004	0.004	0.000
0.7	diff1	0.168	0.291	0.292	0.249	0.166	0.289	0.290	0.255	0.000
	diff2	0.168	0.298	0.328	0.206	0.166	0.289	0.290	0.255	0.000
	sim	0.126	0.266	0.301	0.307	0.125	0.259	0.267	0.349	0.002
	c.i.	0.004	0.006	0.004	0.003	0.003	0.006	0.004	0.007	0.001
1.0	diff1	0.107	0.232	0.311	0.350	0.107	0.228	0.280	0.385	0.001
	diff2	0.109	0.248	0.358	0.285	0.107	0.227	0.280	0.385	0.001
	sim	0.078	0.213	0.324	0.385	0.073	0.189	0.242	0.495	0.001
	c.i.	0.002	0.005	0.008	0.007	0.004	0.005	0.007	0.008	0.002
3.0	diff1	0.047	0.166	0.331	0.456	0.027	0.084	0.147	0.742	0.106
	diff2	0.055	0.192	0.385	0.368	0.027	0.084	0.147	0.742	0.076
	sim	0.054	0.192	0.344	0.411	0.019	0.065	0.113	0.803	0.116
	c.i.	0.004	0.005	0.008	0.006	0.002	0.002	0.004	0.005	0.004
5.0	diff1	0.047	0.167	0.334	0.452	0.016	0.051	0.095	0.838	0.197
	diff2	0.055	0.194	0.389	0.363	0.016	0.052	0.097	0.835	0.136
	sim	0.053	0.189	0.344	0.414	0.012	0.040	0.075	0.873	0.170
	c.i.	0.002	0.005	0.004	0.006	0.001	0.003	0.004	0.005	0.004
10.0	diff1	0.047	0.167	0.335	0.451	0.009	0.032	0.063	0.895	0.257
	diff2	0.054	0.193	0.388	0.364	0.010	0.035	0.069	0.886	0.171
	sim	0.053	0.187	0.352	0.408	0.008	0.029	0.059	0.904	0.201
	c.i.	0.003	0.005	0.004	0.005	0.001	0.001	0.003	0.003	0.005
20.0	diff1	0.047	0.165	0.331	0.457	0.008	0.029	0.059	0.903	0.267
	diff2	0.055	0.195	0.391	0.360	0.009	0.033	0.067	0.890	0.172
	sim	0.054	0.188	0.340	0.418	0.007	0.027	0.053	0.913	0.208
	c.i.	0.003	0.005	0.005	0.007	0.001	0.002	0.002	0.004	0.006

TABLE 6

Comparison of the diffusion results with simulation for $M/H_2/3/N$ Queue ($x_0 = 0$, $\mu = 1.0$, $\rho = 0.95$)

t	N / prob. / method	3				8				
		$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_N(t)$	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_B(t)$	$P_N(t)$
0.3	diff1	0.469	0.334	0.153	0.043	0.466	0.332	0.149	0.051	0.000
	diff2	0.470	0.335	0.152	0.043	0.466	0.332	0.149	0.051	0.000
	sim	0.485	0.352	0.128	0.035	0.487	0.350	0.127	0.036	0.000
	c.i.	0.007	0.006	0.004	0.003	0.007	0.007	0.003	0.003	0.000
0.5	diff1	0.336	0.324	0.219	0.121	0.337	0.322	0.208	0.133	0.000
	diff2	0.339	0.325	0.216	0.120	0.337	0.322	0.208	0.133	0.000
	sim	0.339	0.366	0.205	0.090	0.340	0.365	0.198	0.097	0.000
	c.i.	0.007	0.009	0.005	0.004	0.005	0.005	0.003	0.004	0.000
0.7	diff1	0.256	0.299	0.253	0.193	0.259	0.296	0.232	0.213	0.000
	diff2	0.260	0.301	0.250	0.189	0.259	0.296	0.232	0.213	0.000
	sim	0.254	0.350	0.252	0.144	0.255	0.346	0.237	0.162	0.000
	c.i.	0.003	0.005	0.004	0.005	0.006	0.009	0.005	0.004	0.000
1.0	diff1	0.187	0.268	0.277	0.269	0.190	0.258	0.239	0.313	0.003
	diff2	0.191	0.271	0.275	0.263	0.190	0.258	0.239	0.313	0.003
	sim	0.182	0.320	0.298	0.200	0.182	0.308	0.260	0.250	0.002
	c.i.	0.005	0.004	0.007	0.006	0.004	0.005	0.005	0.005	0.001
3.0	diff1	0.108	0.224	0.298	0.370	0.076	0.144	0.177	0.602	0.060
	diff2	0.111	0.226	0.302	0.361	0.077	0.146	0.179	0.598	0.068
	sim	0.094	0.255	0.341	0.310	0.073	0.176	0.215	0.536	0.038
	c.i.	0.004	0.005	0.006	0.005	0.002	0.004	0.005	0.005	0.002
5.0	diff1	0.107	0.222	0.299	0.372	0.057	0.116	0.150	0.674	0.095
	diff2	0.109	0.225	0.303	0.363	0.058	0.115	0.149	0.677	0.110
	sim	0.090	0.242	0.343	0.325	0.051	0.138	0.178	0.633	0.076
	c.i.	0.003	0.004	0.005	0.006	0.003	0.004	0.004	0.005	0.003
10.0	diff1	0.108	0.223	0.299	0.371	0.049	0.102	0.136	0.712	0.115
	diff2	0.109	0.226	0.302	0.363	0.049	0.100	0.133	0.719	0.137
	sim	0.083	0.246	0.347	0.325	0.039	0.111	0.154	0.695	0.109
	c.i.	0.003	0.005	0.006	0.006	0.002	0.005	0.005	0.006	0.005
20.0	diff1	0.108	0.223	0.298	0.371	0.049	0.102	0.136	0.714	0.111
	diff2	0.110	0.226	0.302	0.363	0.048	0.099	0.132	0.721	0.136
	sim	0.082	0.244	0.348	0.326	0.041	0.106	0.149	0.704	0.116
	c.i.	0.004	0.004	0.006	0.005	0.002	0.003	0.005	0.006	0.004

TABLE 7

Comparison of the diffusion results with simulation for $M/H_2/3/N$ Queue ($x_0 = 0,\ \mu = 1.0,\ \rho = 1.2$)

t	N / prob. / method	3: $P_0(t)$	3: $P_1(t)$	3: $P_2(t)$	3: $P_N(t)$	8: $P_0(t)$	8: $P_1(t)$	8: $P_2(t)$	8: $P_B(t)$	8: $P_N(t)$
0.3	diff1	0.397	0.339	0.194	0.074	0.393	0.335	0.187	0.085	0.000
	diff2	0.397	0.338	0.192	0.072	0.393	0.335	0.187	0.085	0.000
	sim	0.406	0.364	0.170	0.060	0.405	0.364	0.168	0.063	0.000
	c.i.	0.007	0.007	0.004	0.003	0.006	0.006	0.005	0.003	0.000
0.5	diff1	0.265	0.305	0.257	0.173	0.263	0.299	0.237	0.202	0.000
	diff2	0.265	0.304	0.251	0.179	0.263	0.299	0.237	0.202	0.000
	sim	0.258	0.349	0.250	0.144	0.256	0.346	0.239	0.159	0.000
	c.i.	0.005	0.005	0.005	0.003	0.005	0.005	0.005	0.003	0.000
0.7	diff1	0.192	0.269	0.286	0.253	0.190	0.257	0.245	0.309	0.001
	diff2	0.192	0.266	0.276	0.266	0.190	0.257	0.245	0.309	0.001
	sim	0.182	0.313	0.290	0.216	0.180	0.307	0.263	0.251	0.001
	c.i.	0.003	0.005	0.005	0.005	0.003	0.006	0.005	0.005	0.000
1.0	diff1	0.134	0.234	0.305	0.327	0.129	0.206	0.231	0.433	0.009
	diff2	0.133	0.228	0.292	0.348	0.129	0.206	0.231	0.434	0.009
	sim	0.123	0.269	0.319	0.289	0.117	0.251	0.260	0.372	0.004
	c.i.	0.004	0.006	0.006	0.006	0.004	0.005	0.006	0.006	0.001
3.0	diff1	0.080	0.196	0.319	0.404	0.039	0.086	0.126	0.748	0.132
	diff2	0.076	0.186	0.301	0.436	0.039	0.086	0.126	0.749	0.148
	sim	0.059	0.202	0.343	0.395	0.033	0.106	0.154	0.707	0.104
	c.i.	0.002	0.004	0.006	0.005	0.001	0.003	0.003	0.005	0.003
5.0	diff1	0.080	0.197	0.320	0.403	0.026	0.060	0.094	0.821	0.188
	diff2	0.076	0.186	0.303	0.435	0.025	0.059	0.092	0.824	0.217
	sim	0.055	0.189	0.347	0.409	0.021	0.064	0.109	0.806	0.178
	c.i.	0.002	0.004	0.005	0.006	0.002	0.003	0.004	0.005	0.006
10.0	diff1	0.080	0.197	0.320	0.403	0.020	0.050	0.080	0.850	0.211
	diff2	0.075	0.184	0.299	0.441	0.019	0.047	0.077	0.857	0.248
	sim	0.051	0.193	0.343	0.413	0.014	0.047	0.083	0.856	0.225
	c.i.	0.003	0.004	0.004	0.006	0.002	0.003	0.004	0.005	0.006
20.0	diff1	0.076	0.185	0.300	0.440	0.020	0.049	0.080	0.851	0.216
	diff2	0.071	0.172	0.278	0.479	0.019	0.047	0.076	0.858	0.255
	sim	0.052	0.189	0.346	0.413	0.014	0.048	0.080	0.858	0.229
	c.i.	0.003	0.005	0.008	0.006	0.001	0.004	0.005	0.006	0.004

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