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PSEUDO–UMBILICAL SURFACES IN A PSEUDO–RIEMANNIAN SPHERE OR A PSEUDO–HYPERBOLIC SPACE

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1. Introduction

The notion of finite type submanifold was introduced by B.-Y. Chen [1]. A lot of works were done in this field of study by many authors. B.-Y. Chen also extended this notion to pseudo-Riemannian submanifold of pseudo-Euclidean space ([2]). Let E_s^m be the m-dimensional pseudo-Euclidean space with metric tensor of the form

$$\tilde{g} = -\sum_{i=1}^{s} dx_i^2 + \sum_{i=s+1}^{m} dx_i^2$$

where $(x_1, x_2, \cdots, x_m)$ in a rectangular coordinate system in E_s^m. Let M be a connected n-dimensional pseudo-Riemannian submanifold of E_s^{m+1} with signature $(r, n-r)$. We then have the Laplacian Δ of M acting on the space of smooth functions on M. M is said to be k-type if the component functions of position vector x of M in E_s^m can be expressed as a finite sum of eigenfunctions of Δ:

$$\begin{aligned} x &= x_0 + x_{i_1} + x_{i_2} + \cdots + x_{i_k}, \\ \Delta x_{i_j} &= l_{i_j} x_{ij}, l_{i_1} < l_{i_2} < \cdots < l_{i_k} \end{aligned} \tag{1.1}$$

for some natural number k where x_0 is a constant map, $x_{i_1}, \cdots, x_{i_k}$ non constant maps. A k-type submanifold is said to be null k-type if one of $l_{i_1}, \cdots, l_{i_k}$ is zero. Let c be a point of E_s^m and $r > 0$. We put

$$S_s^m(c, r) = \{x \in E_s^{m+1} | \langle x - c, x - c\rangle = r^2\},$$
$$H_s^m(c, r) = \{x \in E_{s+1}^{m+1} | \langle x - c, x - c\rangle = -r^2\},$$

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where $\langle , \rangle$ denotes the indefinite inner product. It is known that $S_s^m(c,r)$ and H_s^m are complete pseudo-Riemannian manifolds of signature $(s, m-s)$ with constant sectional curvatures $1/r^2$ and $-1/r^2$ respectively. $S_s^m(c,r)$ is simply connected if $s < m-1$ and $S_{m-1}^m(c,r)$ is connected and has infinite cyclic fundamental group. $S_m^m(c,r)$ has two simply connected components. $S_s^m(c,r)$ and $H_s^m(c,r)$ are respectively called the pseudo-Riemannian sphere with center c and the pseudo-hyperbolic space with center c. B.-Y. Chen ([2]) studied 1-type compact space like submanifolds of $S_s^m(c,r)$ and $H_s^m(c,r)$ and classified them. A vector X of E_s^{m+1} is called spacelike (timelike or lightlike, respectively) if $\langle X,X\rangle \geq 0$ ($\langle X,X\rangle < 0, \langle X,X\rangle = 0$ and $X \neq 0$). By a *hyperplane section* N of S_s^m (or of H_{s-1}^m) we mean the intersection of S_s^m (or of H_{s-1}^m) and the hyperplane L of E_{s-1}^{m+1}. In particular, the hyperplane section is called the *null hypersection* if the normal vector to the hyperplane is a null vector in E_s^{m+1} (see [6]).

Let M be an n-dimensional pseudo-Riemannian submanifold of an m-dimensional pseudo-Riemannian manifold $\tilde{M}$. Let $e_1, e_2, \cdots, e_n$ be an orthonormal basis for the tangent bundle TM and $e_{n+1}, \cdots, e_m$ an orthonormal normal basis for the normal bundle $T^\perp M$. Since M is pseudo-Riemannian submanifold of $\tilde{M}$, the tangent bundle of $\tilde{M}$ can be decomposed as $T\tilde{M} = TM \oplus T^\perp M$. Let ϕ be a smooth endomorphism of $T_pM, p \in M$. We then define the trace ϕ as follows:

$$tr\phi = \sum_i \epsilon_i \langle \phi e_i, e_i\rangle, \epsilon_i = \langle e_i, e_i\rangle = \pm 1.$$

Thus, the mean curvature vector H of M is given by

$$H = \frac{1}{n}\sum_i \epsilon_i h(e_i, e_i),$$

where h is the second fundamental form of M. M is said to be pseudo-umbilical if $A_H = \rho I$, $\langle H,H\rangle \neq 0$ where A is the weingarten map, ρ a smooth function, and I the identity transformation.

REMARK. ([6]) Null hypersections with nonvanishing mean curvature vector are of infinite type.

In this article, we study 2-type pseudo-umbilical surface in a pseudo-Riemannian sphere or a pseudo-hyperbolic space.

2. Main results

Let $\tilde{M}$ be an isometric immersion of a surface in a pseudo-Riemannian sphere $S_s^m(c,r)$ or a pseudo-hyperbolic space $H_{s-1}^m(c,r)$ in E_s^{m+1} and let $x : M \longrightarrow \tilde{M}$ be an isometric immersion of a surface M into $\tilde{M}$ with respect to induced metric. We can identify x with the position vector. We may assume that c is the origin of E_s^{m+1} and $r = 1$. We simply call M pseudo-spherical if $x(M)$ lies in S_s^m or H_{s-1}^m. Let H, A, h, D be the mean curvature vector, the Weingarten map, the second fundamental form and the normal connection of M in E_s^{m+1} and H', A', h', D' the corresponding terms of M in $\tilde{M}$.

Since x is the unit vector normal to $\tilde{M}$, $Dx = 0$. We also denote by $\tilde{\nabla}$ and ∇ the Levi-Civita connections of E_s^{m+1} and M, respectively. Then we have $H = H' - \epsilon x$, where $\epsilon = 1$ if $\tilde{M} = S_s^m$ and $\epsilon = -1$ if $\tilde{M} = H_{s-1}^m$. It is easily seen that $A'_{H'} = A_{H'}$ and $D'H' = DH'$. Thus, one can compute

$$(2.1) \qquad \Delta H = \Delta^{D'} H' + 2\epsilon H' + \sum_i \epsilon_i h'(A_{H'} e_i, e_i) - (\epsilon tr A_{H'} + 2)x + tr\tilde{\nabla} A_{H'},$$

where $\Delta^{D'} H' = -\sum \epsilon_i (D'_{e_i} D'_{e_i} H' - D'_{\nabla_{e_i} e_i} H')$ and $tr\tilde{\nabla} A_{H'} = tr\nabla A_{H'} + trA_{DH'} = \sum_i \epsilon_i\{(\nabla_{e_i} A_{H'})e_i + A_{D_{e_i} H'} e_i\}$ for some orthonormal basis e_1, e_2 of TM. Suppose M is pseudo-umbilical in $\tilde{M}$. Then, $A'_{H'} = \epsilon'(\alpha')^2 I$, where $\epsilon' = \text{sign}\langle H', H'\rangle$ and $|\langle H', H'\rangle|^{1/2} = \alpha'$, that is, α' is the mean curvature of M in $\tilde{M}$. We can easily see that M is pseudo-umbilical in $\tilde{M}$ if and only if so is M in E_s^{m+1}. We now suppose M is of 2-type. Then, the position vector x of M in E_s^{m+1} can be written as $x = x_0 + x_p + x_q$, $\Delta x_p = \lambda_p x_p$, $\Delta x_q = \lambda_q x_q$. Since $\Delta x = -2H$, it follows that

$$(2.2) \qquad \Delta H = bH + c(x - x_0), \quad b = \lambda_p + \lambda_q, \quad c = \frac{\lambda_p \lambda_q}{2}.$$

Since x is a unit normal vector field to M, (2.1) and (2.2) give

$$-\epsilon(\epsilon tr A_{H'} + 2) = -b + c\epsilon - c\langle x, x_0\rangle,$$

or

$$c\langle x, x_0\rangle = \epsilon(2 + \epsilon tr A_{H'} + c - \epsilon b). \tag{2.3}$$

Let X be any tangent vector field to M. From (2.1) we get

$$\langle \Delta H, X\rangle = \langle tr\bar{\nabla} A_{H'}, X\rangle.$$

On the other hand, (2.2) yields

$$\langle \Delta H, X\rangle = -c\langle x_0, X\rangle.$$

Differentiating (2.3) covariantly, we get

$$c\langle X, x_0\rangle = X(tr A_{H'}),$$

from which,

$$tr\bar{\nabla} A_{H'} = -2\text{grad}\langle H', H'\rangle,$$

or,

$$c(x_0)^T = 2\text{grad}\langle H', H'\rangle, \tag{2.4}$$

where $(x_0)^T$ denotes the tangential component of x_0.
Quite a similar work to get (4.7) in pp. 270, [1] gives

$$\Delta H = \Delta^D H + \text{grad}\langle H, H\rangle + 2tr A_{DH} + \sum_r \epsilon_r tr(A_H A_r) e_r, \tag{2.5}$$

where $A_r = A_{e_r}$, $\{e_r\}$ an orthonormal normal frame and $\langle e_r, e_r\rangle = \epsilon_r$. Since $Dx = 0$ and $DH' = D'H'$, the fact that M is pseudo-umbilical together with (2.5) yields

$$\Delta H = \Delta^{D'} H' + \text{grad}\langle H', H'\rangle + 2tr A_{D'H'} + 2\langle H, H\rangle H. \tag{2.6}$$

Thus, (2.2) and (2.6) produce

$$\begin{aligned}&\Delta^{D'} H' + \text{grad}\langle H', H'\rangle + 2tr A_{D'H'} + 2\langle H, H\rangle(H' - \epsilon x)\\ &\quad = b(H' - \epsilon x) + c(x - x_0),\end{aligned}$$

or,

$$\Delta^D H + 2\langle H, H\rangle H + \text{grad}\langle H', H'\rangle + 2tr A_{DH} = bH + c(x - x_0). \tag{2.7}$$

It follows that

$$-c(x_0)^T = \text{grad}\langle H', H'\rangle + 2tr A_{D'H'}.$$

Combining the last equation and (2.4), we see

$$3\text{grad}\langle H', H'\rangle - 2tr A_{D'H'} = 0. \tag{2.8}$$

LEMMA 1. *Let M be a pseudo-umbilical surface in $\tilde{M}$. Then, we have*

$$trA_{DH} = 0.$$

Consequently, $trA_{D'H'} = 0$.

Proof. Let $\{e_1, e_2\}$ be a local orthonormal frame over M. Then,

$$\begin{aligned}
trA_{DH} &= \sum_i \epsilon_i A_{D_{e_i}H} e_i = \sum_{i,j} \epsilon_i\epsilon_j \langle A_{D_{e_i}H} e_i, e_j\rangle e_j \\
&= \sum_{i,j} \epsilon_i\epsilon_j \langle h(e_i, e_j), D_{e_i}H\rangle e_j \\
&= \sum_{i,j} \epsilon_i\epsilon_j \{e_i\langle H, h(e_i, e_j)\rangle - \langle H, D_{e_i}h(e_i, e_j)\rangle\} e_j \\
&= \sum_{i,j} \epsilon_i\epsilon_j \{e_i\langle A_H e_i, e_j\rangle - \langle H, (\bar{\nabla}_{e_i}h)(e_i, e_j) + h(\nabla_{e_i}e_j, e_i) \\
&\quad + h(\nabla_{e_i}e_i, e_j)\rangle\} e_j \\
&= \sum_{i,j} \epsilon_i\epsilon_j \{e_i(\langle H, H\rangle)\langle e_i, e_j\rangle - \langle H, (\bar{\nabla}_{e_j}h)(e_i, e_i)\rangle \\
&\quad - \langle A_H e_i, \nabla_{e_i}e_j\rangle - \langle A_H e_j, \nabla_{e_i}e_i\rangle\} e_j \\
&= \sum_{i,j} \epsilon_i\epsilon_j \{e_i(\langle H, H\rangle)\langle e_i, e_j\rangle - \langle H, D_{e_j}h(e_i, e_i)\rangle \\
&\quad + 2\langle H, h(\nabla_{e_j}e_i, e_i)\rangle\} e_j \\
&= 0 \quad \text{since} \quad A_H = \langle H, H\rangle I. \quad ||||
\end{aligned}$$

Therefore, (2.8) implies $\langle H', H'\rangle$ is constant on M and so is $\langle H, H\rangle$. From (2.4), we can have either $c = 0$ or $(x_0)^T = 0$. If $c = 0$, then M is of null 2-type. It follows from (2.7)

$$\Delta^D H = 0 \quad \text{and} \quad b = 2\langle H, H\rangle. \tag{2.9}$$

However, M cannot have a parallel mean curvature vector due to Lemma 2 in [2]. If $c \neq 0$, then $(x_0)^T = 0$, that is, x_0 is a constant normal vector field. Suppose $x_0 \neq 0$. (2.3) implies that x lies in a hyperplane in E^m_s and we may regard M as a pseudo-umbilical surface of hyperplane sections. Thus, M can be considered as an open portion of S^2_t (or H^2_{t-1}) or null hypersection, which is not of 2-type. Therefore, x_0 must be the zero vector. Hence, we have

THEOREM 2. *Let M be a pseudo-umbilical 2-type surface in S^m_s or H^m_{s-1}. Then, M with non-parallel mean curvature vector has constant mean curvature and one of the following occurs :*

(1) If M is of null 2-type, then M has a harmonic mean curvature vector with respect to normal connection.

(2) If M is of non-null 2-type, then x_0 is the zero vector.

COROLLARY 3. *If M is a compact spacelike pseudo-umbilical non-null 2-type surface in S^m_s or H^m_{s-1}, then M is mass-symmetric.*

COROLLARY 4. *There is no Lorentzian pseudo-umbilical 2-type surface with parallel mean curvature vector in S^m_s or H^m_{s-1}.*

THEOREM 5. *Let M be a pseudo-umbilical 2-type surface in S^m_s or H^m_{s-1}. Then m must be bigger than 4.*

Proof. Let M be pseudo-umbilical 2-type surface of S^4_s or H^4_{s-1} with mean curvature vector H'. We denote by $\epsilon'\beta^2 = \langle H', H'\rangle$ where $\epsilon' = \operatorname{sign}\langle H', H'\rangle$ and $\beta = |\langle H', H'\rangle|^{\frac{1}{2}} > 0$. We choose an orthonormal basis e_1, e_2, e_3, e_4 of TS^4_s or TH^4_{s-1} such that e_1, e_2 are tangent to M and e_3, e_4 normal to M with $A_3 = \epsilon'\beta I$ and $trA_4 = 0$. We denote by w^1, w^2, w^3, w^4 the dual 1-forms. Then the connection forms w^B_A are given by

$$de_A = \sum_B w^B_A \otimes e_B, \quad w^B_A + \epsilon_A\epsilon_B w^A_B = 0,$$

where $\epsilon_A = \langle e_A, e_A\rangle$, $A = 1, 2, 3, 4$. We obtain the following from the above equations

$$dw^A = \sum_{C=1}^{4} w^C \wedge w^A_C, \tag{2.10}$$

$$dw^B_A = \sum_{C=1}^{4} w^C_A \wedge w^B_C + \Omega^B_A, \tag{2.11}$$

where $\Omega^B_A = \frac{1}{2}\sum_{C,D} R^B_{ACD} w^C \wedge w^D$, $\quad R^B_{ACD} = \epsilon_B\langle R(e_C, e_D)e_A, e_B\rangle$ and R is the Riemann-Christoffel curvature tensor of S^4_s or H^4_{s-1}.

We put

$$h = \sum_{i,j,t} h^t_{ij} w^i \otimes w^j \otimes e_t, \tag{2.12}$$

where $i, j \in \{1, 2\}$ and $t \in \{3, 4\}$. Then we have

$$w^t_i = \sum_j h^t_{ij} w^j, \quad \langle A_t e_i, e_j \rangle = \epsilon_t h^t_{ij}, \tag{2.13}$$

where $A_t = A_{e_t}$. Since $A_3 = \epsilon_3 \beta I$,

$$w^3_i = \beta \epsilon_i w^i \quad (i = 1, 2). \tag{2.14}$$

Making use of (2.10), we get

$$dw^3_i = \beta \epsilon_i dw^i = \beta \epsilon_i \sum_{j=1}^{2} w^j \wedge w^i_j \tag{2.15}$$

because β is constant. On the other hand, if we use (2.11) and the fact that S^4_S or H^4_{S-1} is of constant curvature, then (2.14) implies

(2.16)

$$\begin{aligned} dw^3_i &= \sum_{j=1}^{2} w^j_i \wedge w^3_j + w^4_i \wedge w^3_4 \\ &= \epsilon_j \beta \sum_{j=1}^{2} w^j_i \wedge w^j + w^4_i \wedge w^3_4 = \epsilon_i \beta \sum_{j=1}^{2} w^j \wedge w^i_j + w^4_i \wedge w^3_4. \end{aligned}$$

It follows that

$$w^4_i \wedge w^3_4 = 0 \quad (i = 1, 2). \tag{2.17}$$

We now split the cases :

Case (1). Suppose M is timelike or spacelike. Without loss of generality, we may assume M is spacelike. Then, (2.16) implies Then, (2.17) implies

$$w^4_i(e_j) w^3_4(e_k) - w^4_i(e_k) w^3_4(e_j) = 0.$$

Since $w_i^4(e_j) = h_{ij}^4$, we get

$$(2.18) \qquad h_{ij}^4 w_4^3(e_k) - h_{ik}^4 w_4^3(e_j) = 0.$$

Since $trA_4 = 0$, (2.18) yields

$$(2.19) \qquad \sum_i h_{ik}^4 w_4^3(e_i) = 0.$$

Let $M_0 = \{p \in M | w_4^3(p) \neq 0\}$. Suppose $M_0 \neq \phi$. Then, (2.17) and (2.19) implies

$$w_i^4 = 0$$

on M_0. (2.11) and the last equation give

$$0 = dw_i^4 = \sum_B w_i^B \wedge w_B^4 = w_i^3 \wedge w_3^4 = \beta w^i \wedge w_3^4,$$

from which,

$$w_3^4 \wedge w^i = 0 \quad \text{on} \quad M_0.$$

This implies $w_3^4 = 0$ on M_0, which is a contradiction. Therefore, $M_0 = \phi$, that is, w_3^4 vanishes on M. Hence, e_3 is parallel in the normal bundle, that is, the mean curvature vector is parallel. By applying Lemma 2 and Proposition 1 in [2] again, we see that M is minimal in hyperplane sections which is a contradiction (see [6]).

Case (2). Let M be Lorentzian. Then, the Weingarten map A_4 has one of the following forms ([4]):

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}; \quad \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

for some functions λ, μ, a and b, where the first and the last representations are induced by an orthonormal frame and the second one is obtained by an pseudo-orthonormal frame $\{e_1, e_2\}$ on M satisfying $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0$, $\langle e_1, e_2\rangle = 1$. Since M is pseudo-umbilical, we can choose an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ such that e_1 and e_2 are tangent to M, $A_3 = \epsilon_3 \beta I$ and A_4 has one of the following :

$$\begin{pmatrix} -\lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} -1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}, \quad \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}.$$

Suppose $A_4 = \begin{pmatrix} -\lambda & 0 \\ 0 & \lambda \end{pmatrix}$. If we assume $\epsilon_1 = \langle e_1, e_1 \rangle = -1$ and $\epsilon_2 = \langle e_2, e_2 \rangle = 1$, then we get

$$w_i^4 = (-1)^i \epsilon_i \epsilon_4 \lambda w^i, \tag{2.20}$$

which together with (2.17) gives

$$\lambda w^i \wedge w_4^3 = 0 \quad (i = 1, 2).$$

Since e_3 cannot be parallel, λ vanishes on M. Therefore, M is totally umbilical in S_3^4 or H_{s-1}^4, which is not of 2-type.

Suppose $A_4 = \begin{pmatrix} -1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$. Then, $w_1^4 = w_2^4 = \frac{\epsilon_4}{2}(w^1 + w^2)$. It follows that

$$w_3^4(e_1) = w_3^4(e_2) \tag{2.21}$$

because of (2.17). The structure equation combined with (2.21) implies

$$dw_3^4 = 0.$$

Then, the distribution

$$\mathcal{D} = \{X \in TM | w_3^4(X) = 0\}$$

is integrable and parallel because of (2.21), where TM is the set of vector fields on M. The integral submanifold of $\mathcal{D}$ is a null curve γ. It is easy to see that γ is a geodesic in M. Let $\dot{\gamma} = X$ and let $\{X, Y\}$ be a pseudo-orthonormal frame such that $\langle X, X \rangle = \langle Y, Y \rangle = 0$ and $\langle X, Y \rangle = 1$. Then, Y generates the complementary distribution $\mathcal{D}^\perp$ such that $TM = \mathcal{D} \oplus \mathcal{D}^\perp$. $\mathcal{D}^\perp$ is also integrable and parallel. The integral submanifold of $\mathcal{D}^\perp$ is also a null curve $\tilde{\gamma}$.

On the other hand, we can see that γ and $\tilde{\gamma}$ are totally geodesic in S_s^4 or H_{s-1}^4 and they are null straight lines in E_s^5 (see [7], pp 112-113). Thus, M is an open portion of Lorentz plane E_1^2 that is not pseudo-umbilical in S_s^4 or H_{s-1}^4.

If $A_4 = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$, then

$$w_1^4 = -\epsilon_4 b w^2, \quad w_2^4 = -\epsilon_4 b w^1.$$

The relationships above and (2.17) yield $w_4^3 = 0$, which is contrary to our assumption. Thus the theorem is proved. ||||

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