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A TIGHTNESS THEOREM FOR PRODUCT PARTIAL SUM PROCESSES INDEXED BY SETS

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1. Introduction

Let N denote the set of positive integers. Fix $d_1, d_2 \in \mathbf{N}$ with $d = d_1 + d_2$. Let X and Y be real random variables and let $\{X_{\mathbf{i}} : \mathbf{i} \in \mathbf{N}^{d_1}\}$ and $\{Y_{\mathbf{j}} : \mathbf{j} \in \mathbf{N}^{d_2}\}$ be independent families of independent identically distributed random variables with $\mathcal{L}(X) = \mathcal{L}(X_{\mathbf{i}})$ and $\mathcal{L}(Y) = \mathcal{L}(Y_{\mathbf{j}})$, where $\mathcal{L}(\cdot)$ denote the law of $\cdot$.

We define the product partial sum process T_n corresponding to $\{X_{\mathbf{i}}\}$ and $\{Y_{\mathbf{j}}\}$ indexed by subsets of the d-dimensional unit cube $\mathbf{I}^d$ by

$$T_n(X, Y, A) := \sum_{|\mathbf{i}| \le n, |\mathbf{j}| \le n} X_{\mathbf{i}} Y_{\mathbf{j}} \delta_{(\mathbf{i}/n, \mathbf{j}/n)}(A), \qquad A \subset \mathbf{I}^d,$$

where $(\mathbf{i}/n, \mathbf{j}/n) = (i_1/n, i_2/n, \cdots, i_{d_1}/n, j_1/n, j_2/n, \cdots, j_{d_2}/n)$ and $\delta_{(\mathbf{i}/n, \mathbf{j}/n)}(A) = 1$ or 0 depending on $(\mathbf{i}/n, \mathbf{j}/n) \in A$ or not with i's and j's integers. For product partial sum processes T_n, laws of large number results have been shown to hold (for example, [6], [7] under some metric entropy condition). It is therefore quite natural to study weak convergence problems (Central Limit Theorem) for these product processes. We say that random elements Y_n, Y taking values in $l^\infty(\mathcal{F})$ satisfies CLT iff the finite dimensional distributions of Y_n converge in law to those of Y and there exists a psedometric ρ on $\mathcal{F}$ such that $(\mathcal{F}, \rho)$ is totally bounded and

$$\limsup_{\delta \to 0} \limsup_{n \to \infty} P^* \Big(\sup_{\rho(f,g) \le \delta} |Y_n(f) - Y_n(g)| > \epsilon \Big) = 0$$

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for all $\epsilon > 0$.

To ensure the weak convergence of T_n, first we need a limiting process, product Brownian measure, which is constructed in [8] as follows; Let $(Z_1, \mathcal{A}_1)$ and $(Z_2, \mathcal{A}_2)$ represent two independent Brownian measures with $\mathcal{A}_i \subset \mathcal{B}_i \cap \mathbf{I}^{d_i}$. Define $Z(A_1 \times A_2) = Z_1(A_1)Z_2(A_2)$ on the field generated by $\mathcal{A}_1 \times \mathcal{A}_2$. Then, the domain of Z can be extended beyond $\mathcal{A}_1 \times \mathcal{A}_2$ to as large a subfamily $\mathcal{A}$ of the $\sigma-$field $\sigma(\mathcal{A}_1 \times \mathcal{A}_2)$, so that Z on $\mathcal{A}$ is uniformly continuous with respect to the symmetric difference pseudo-metric $d(A, B) = |A \triangle B|$. Next we need to smooth T_n(for the reason, see [1]) as follow: Define, for $A \in \mathcal{A}$,

$$S_n(A) := n^{-d/2} \sum_{|\mathbf{i}| \leq n} \sum_{|\mathbf{j}| \leq n} X_{\mathbf{i}} Y_{\mathbf{j}} |nA \cap C_{\mathbf{ij}}|,$$

the normalized smoothed product partial sum process of T_n, where $C_{\mathbf{ij}}$ is d-dimensional unit cube whose Lebesgue measure is 1 and the upper right corner has a coordinate $(\mathbf{i}, \mathbf{j})$ with $\mathbf{i} \in \mathbf{N}^{d_1}$ and $\mathbf{j} \in \mathbf{N}^{d_2}$. Finally we impose some restrictions on the index family $\mathcal{A} \subset \mathcal{B}(\mathbf{I}^d)$ in terms of entropy condition. Our entropy condition is the same as the one in [8], which is conjectured there. Throughout the paper, assume that $X_{\mathbf{i}}$'s and $Y_{\mathbf{j}}$'s are sub-Gaussian random variables. That is, there exists some constant M and γ depending on $X_{\mathbf{i}}$ such that $P(|X_{\mathbf{i}}| > x) \leq M \int_x^\infty e^{-\gamma t^2} dt$. We tried to prove the convergence of finite dimensional distributions of T_n but unfortunately we cannot get satisfiable results yet. In this paper thus we proves only a tightness theorem for product partial sum processes indexed by subsets of $[0, 1]^d$ and based on i.i.d. sub-Gaussian random variables.

The outline of this paper is as follow. In Section 2 we derive an exponential probability bound for S_n using conditioning and the Hans on-Wright inequality [5], which is comparable to the bounds in [8], and we apply this bound to prove a tightness result in Section 3.

2. Bounds for S_n

Let us begin with the Hanson-Wright inequality which is central in deriving probability bound for S_n. Suppose a_{ij}, $i, j \in \mathbf{N}$ are real numbers such that $a_{ij} = a_{ji}$ and $\Lambda^2 := \sum_{i,j} a_{ij}^2 < \infty$. Let $\mathbf{A}$ denote the matrix $(|a_{ij}|)$ and let $\|\mathbf{A}\|_2$ be the norm of $\mathbf{A}$ considered as an operator on $l^2(\mathbf{A})$. Define $S := \sum_{ij} a_{ij}(X_iX_j - EX_iX_j)$, where EZ is the expectation of a random variable Z. Under the assumption stated above, the Hanson-Wright inequality is as follows.

LEMMA 2.1. ([8]) *For every $\varepsilon > 0$, there exist constants C_1 and C_2 depending on M and γ(but not on $\mathbf{A}$) such that*

$$P(S \geq \varepsilon) \leq \exp(-\min\{C_1\varepsilon/\|\mathbf{A}\|_2, C_2\varepsilon^2/\Lambda^2\}).$$

THEOREM 2.2. *For any $\eta > 0$ and for some constants K_1 and K_2, we have*

$$P(S_n(A) > \eta) \leq \exp(-K_1\eta/|A|^{1/2}) + \exp(-K_2\eta^{4/3}/|A|^{2/3})$$

where $|A|$ denotes the Lebesgue measure of $A \in \mathcal{A}$.

Proof. Let $\mathcal{F}_n = \sigma(Y_{\mathbf{i}} : |\mathbf{i}| \leq n)$ be the σ- algebra generated by $Y_{\mathbf{i}}$ and $S_n := S_n(A)$. Since $X_{\mathbf{i}}$'s are independent sub-Gaussian of $Y_{\mathbf{j}}$'s, for any $\lambda > 0$,

$$E(e^{\lambda S_n}|\mathcal{F}_n) \leq \exp(c\lambda^2 n^{-d} \sum_{|\mathbf{i}|\leq n} (\sum_{|\mathbf{j}|\leq n} Y_{\mathbf{j}}|nA \cap C_{\mathbf{ij}}|)^2)$$

where c is a positive constant only depending on a sub-Gaussian random variable $X_{\mathbf{i}}$. Now

$$\begin{aligned} n^{-d} &\sum_{|\mathbf{i}|\leq n} \Big(\sum_{|\mathbf{j}|\leq n} Y_{\mathbf{j}}|nA \cap C_{\mathbf{j}}|\Big)^2 \\ &= n^{-d} \sum_{|\mathbf{i}|\leq n} \Big(\sum_{|\mathbf{j}|\leq n} Y_{\mathbf{j}}|nA \cap C_{\mathbf{ij}}|\Big)\Big(\sum_{|\mathbf{k}|\leq n} Y_{\mathbf{k}}|nA \cap C_{\mathbf{ik}}|\Big) \\ &= n^{-d} \sum_{|\mathbf{j}|\leq n} \sum_{|\mathbf{k}\leq n} Y_{\mathbf{j}}Y_{\mathbf{k}} \sum_{|\mathbf{i}|\leq n} |nA \cap C_{\mathbf{ij}}||nA \cap C_{\mathbf{ik}}| \\ &:= \quad Q_n. \end{aligned}$$

Set $a_{\mathbf{jk}} = n^{-d}\sum_{|\mathbf{i}|\leq n} |nA \cap C_{\mathbf{ij}}||nA \cap C_{\mathbf{ik}}|$. Then $(a_{\mathbf{jk}})_{|\mathbf{j}|\leq n, |\mathbf{k}|\leq n}$ is a symmetric matrix and $\Lambda_n^2 := \sum_{|\mathbf{j}|\leq n, |\mathbf{k}|\leq n}(a_{\mathbf{jk}})^2 \leq 1 < \infty$. Since $Y_{\mathbf{j}}$'s are sub-Gaussian, there exists some constant M and γ depending on $Y_{\mathbf{j}}$ such that

$$P(|Y_{\mathbf{j}}| > x) \le M \int_x^\infty e^{-\gamma t^2} dt.$$

Applying the Hanson-Wright inequality we have, $E(e^{\theta Q_n}) \le e^{c_1 \theta^2 \Lambda_n^2}$ for $0 < \theta \le \tau/\Lambda_n$ where c_1 is a positive constant only depending on $Y_{\mathbf{j}}$ and not on n, and τ is constant only depending on $Y_{\mathbf{j}}(M, \gamma)$.

Now look into Λ_n^2. Since $(\sum_{i=1}^n x_i y_i)^2 \le (\sum_{i=1}^n x_i^2)(\sum_{i=1}^n y_i^2)$,

$$\begin{aligned}
\Lambda_n^2 &= n^{-2d} \sum_{|\mathbf{j}| \le n} \sum_{|\mathbf{k}| \le n} \Big(\sum_{|\mathbf{i}| \le n} |nA \cap C_{\mathbf{ij}}||nA \cap C_{\mathbf{ik}}| \Big)^2 \\
&\le n^{-2d} \sum_{|\mathbf{j}| \le n} \sum_{|\mathbf{k}| \le n} \sum_{|\mathbf{i}| \le n} |nA \cap C_{\mathbf{ij}}|^2 \sum_{|\mathbf{l}| \le n} |nA \cap C_{\mathbf{lk}}|^2 \\
&= n^{-2d} \Big(\sum_{|\mathbf{i}| \le n} \sum_{|\mathbf{j}| \le n} |nA \cap C_{\mathbf{ij}}|^2 \Big) \Big(\sum_{|\mathbf{k}| \le n} \sum_{|\mathbf{l}| \le n} |nA \cap C_{\mathbf{lk}}|^2 \Big) \\
&= \Big(n^{-d} \Big(\sum_{|\mathbf{i}| \le n} \sum_{|\mathbf{j}| \le n} |nA \cap C_{\mathbf{ij}}|^2 \Big) \Big)^2
\end{aligned}$$

where we used Hőlder's inequality. So that

$$\Lambda_n \le T_n := n^{-d} \sum_{|\mathbf{i}| \le n} \sum_{|\mathbf{j}| \le n} |nA \cap C_{\mathbf{ij}}|^2 \le |A|.$$

Hence

(2.1)

$$E(\exp(\lambda S_n)) = E[E(\exp(\lambda S_n)|\mathcal{F}_n)]$$

$$\le E\left\{ \exp\left[c\lambda^2 n^{-d} \Big\{ \sum_{|\mathbf{i}| \le n} \big(\sum_{|\mathbf{j}| \le n} Y_{\mathbf{j}} |nA \cap C_{\mathbf{ij}}| \big)^2 \Big\} \right] \right\} \le \exp(c_1 c^2 \lambda^4 \Lambda_n^2).$$

To get an exponential bound for S_n, apply Chebyschev's inequality to (2.1),

$$P(S_n > \eta) \le e^{-\lambda\eta + c_2 \lambda^4 \Lambda_n^2},$$

where $c_2 = c_1 c^2$ and $\lambda \in [-(\tau/c\Lambda_n)^{1/2}, (\tau/c\Lambda_n)^{1/2}]$.

Let $\phi(\lambda) = -\lambda\eta + c_2\lambda^4\Lambda_n^2$. Then $\phi(\lambda)$ has a minimum value at $\lambda_n^{(1)} = [\eta/4c_2\Lambda_n^2]^{1/3}$. Let $\lambda_n = \min\{(\tau/c\Lambda_n)^{1/2}, \lambda_n^{(1)}\}$. Then

$$P(S_n > \eta) \le \exp(-K_1\eta/\Lambda_n^{1/2}) + \exp(-K_2\eta^{4/3}/\Lambda_n^{2/3}), \tag{2.2}$$

where $K_1 = 3\tau^{1/2}/4c^{1/2}$ and $K_2 = 3/c^{4/3}c_2^{1/3}$.

Since $\Lambda_n \le |A|$, (2.2) becomes

$$P(S_n > \eta) \le \exp(-K_1\eta/|A|^{1/2}) + \exp(-K_2\eta^{4/3}/|A|^{2/3})$$

which is independent of n.

3. Main theorem

Now we are ready to prove a tightness result for smoothed product partial sum processes. Define the pseudometric d_λ on $\mathcal{A}$ by $d_\lambda(A,B) = \lambda(A\triangle B) = |A\triangle B|$ where λ and $|\cdot|$ are both used to denote Lebesgue measure. We assume that *with respect to d_λ, $\mathcal{A}$ is totally bounded with inclusion and has a convergent entropy integral.* That is, first, for every $\varepsilon > 0$ there exists a finite collection (called an ε−net) $\mathcal{A}(\varepsilon)$ of measurable sets such that $A \in \mathcal{A}$ implies $A_{(1)} \subset A \subset A^{(2)}$ in $\mathcal{A}(\varepsilon)$, and $d_\lambda(A_{(1)}, A^{(2)}) \le \varepsilon$ for some $A_{(1)}, A^{(2)}$ in $\mathcal{A}(\varepsilon)$. Second, the number of pairs $A_{(1)}, A^{(2)}$ in $\mathcal{A}(\varepsilon)$, which we assume to be the minimum possible and which we denote by

$$\begin{aligned} N_I(\varepsilon, \mathcal{A}, d_\lambda) := \ & \min\{k \ge 1 : \text{there exist measurable sets} \\ & A_{i(1)}, A_i^{(2)}, 1 \le i \le k \\ & \text{such that for every } A \in \mathcal{A} \text{ there is some } i \\ & \text{such that } |A_i^{(2)} \setminus A_{i(1)}| \le \varepsilon \text{ and } A_{i(1)} \subset A \subset A_i^{(2)} \end{aligned}$$

satisfies

$$\int_0^1 \varepsilon^{-1/2} H(\varepsilon)d\varepsilon < \infty. \tag{3.1}$$

or, equivalently, for any $\beta \in (0,1)$

$$(3.2) \qquad \sum_{k \geq 0} \beta^{k/2} H(\beta^{k+1}) < \infty.$$

where $H(\varepsilon) = \log N_I(\varepsilon, \mathcal{A}, d_\lambda)$. Define the *exponent of metric entropy of* $\mathcal{A}$, denoted r, by $r := \inf\{s > 0 : H(\varepsilon) = O(\varepsilon^{-s}) \quad \text{as} \quad \varepsilon \to 0\}$. If $r < 1/2$, then (3.1) holds.

REMARK 3.1. Examples of index families which satisfy our metric entropy assumptions include the following. Let $J(\alpha, d, M)$, for $\alpha > 0, M > 0$, denote the class of sets introduced in [2], whose boundaries are images of α-differentiable mappings of the $(d-1)$-sphere into $\mathbf{I}^d$, with all derivatives of orders up to α uniformly bounded by M. Then, $r = (d-1)/\alpha$; cf. [2]. A related family of sets with α -smooth boundaries, denoted $\mathcal{R}(\alpha, d, M)$, was proposed by [9] and shown there to satisfy $r = (d-1)/\alpha$ as well. Some examples of small classes of sets are $\mathcal{I}^d$, the set of intervals on lower orthants; $\mathcal{P}^{d,m}$, the family of all polygonal regions in $\mathbf{I}^d$ with no more than m vertices; and $\mathcal{E}^d$, the set of all ellipsoidal regions in $\mathbf{I}^d$. For all of these, $r = 0$; see [3] for $\mathcal{P}^{d,m}$, and [4] for $\mathcal{E}^d$. Another important classof sets which includes the last three examples are Vapnik-Cervonenkis classes. For these, it is true that $N(\varepsilon, \mathcal{A}, d_\lambda) \leq C\varepsilon^{-v}$ for some C and $v > 0$, where N is the (usual) metric entropy, like N_I but without the requirement of inclusion.

THEOREM 3.2. (Tightness Theorem) *If $r < 3/5$, then, under* (3.1) *or* (3.2), *and for any $\eta > 0$, we have*

$$\limsup_{k \to \infty} \limsup_{n \to \infty} P(\sup_{A \in \mathcal{A}} |S_n(A) - S_n(A_k)| > \eta) = 0.$$

Proof. Fix $\beta \in (0,1)$ and let $\delta_k = \beta^k$ for any $k \geq 0$. Let $A \in \mathcal{A}$, and let A_k and A^k denote the inner and the outer δ_k approximations to A. Let $\eta > 0$ be fixed and let $\eta_k = c'\beta^{(k+1)/2} H(\beta^{k+1})$, where c' will be chosen later. Let k_0 and k_n be chosen such that

$$k_n > k_0, \qquad \sum_{k \geq k_0} \eta_k < \eta/2, \quad \text{and} \quad n^{d/2} \beta^{2k_n/5} < \eta/2.$$

Now let $R_n(\triangle_{k_n}(A)) := n^{-d/2} \sum_{|\mathbf{i}|\le n} \sum_{|\mathbf{j}|\le n} |X_{\mathbf{i}}||Y_{\mathbf{j}}||n(A^{k_n} \setminus A_{k_n}) \cap C_{\mathbf{ij}}|$. Then

$$\{\omega | R_n(\triangle_{k_n}(A)) > \eta/2, \quad |X_{\mathbf{i}}| \le \beta^{-3k_n/10} \text{and} \\ |Y_{\mathbf{j}}| \le \beta^{-3k_n/10} \quad |\mathbf{i}|, |\mathbf{j}| \le n\} = \emptyset.$$

Hence, after separating into two obvious part, we have

$$\begin{aligned} P^*\Big(\sup_{A_{k_n} \subset A \subset A^{k_n}} &|S_n(A) - S_n(A_{k_n})| > \eta/2\Big) \\ &\le P(|X_{\mathbf{i}}| > \beta^{-3k_n/10} \quad \text{or} \quad |Y_{\mathbf{j}}| > \beta^{-3k_n/10}, \quad |\mathbf{i}|, |\mathbf{j}| \le n) \\ &\le n^{d_1} P(|X_{\mathbf{i}}| > \beta^{-3k_n/10}) + n^{d_2} P(|Y_{\mathbf{j}}| > \beta^{-3k_n/10}) \\ &< 2n^{d_1} \exp(-\beta^{-3k_n/5}/2\theta_1) + 2n^{d_2} \exp(-\beta^{-3k_n/5}/2\theta_2) \end{aligned}$$

where θ_1 and θ_2 are the parameters associated with sub-Gaussian random variables X and Y, and P^* denotes the outer measure induced by P.

By the standard chaining argument, we have

$$|S_n(A_{k_0}) - S_n(A)| \le 2 \sum_{k_0 \le k \le k_n} |S_n(A_k \setminus A_{k+1})| + R_n(\triangle_{k_n}(A)).$$

And

(3.3)
$$\begin{aligned} &P(\sup_{A \in \mathcal{A}} |S_n(A) - S_n(A_{k_0})| > \eta) \\ \le &\sum_{k_0 \le k \le k_n} 2P(|S_n(A_k \setminus A_{k+1})| > \eta_k \quad \text{for some} \quad A \in \mathcal{A}) \\ &+ P(R_n(\triangle_{k_n}(A)) > \eta/2 \quad \text{for some} \quad A \in \mathcal{A}). \end{aligned}$$

Now, by Theorem 2.2,
$P(|S_n(A_k \setminus A_{k+1})| > \eta_k) < 4\exp(-c_1 K_1 H(\beta^{k-1}))$. Let c' be such that

$K_1 c' \geq 3$ with K_1 appearing in Theorem 2.2. Then,

$$\begin{aligned}
&\leq 4 \sum_{k_0 \leq k \leq k_n} \exp(2H(\beta^{k+1}))\exp(-3H(\beta^{k+1}) \\
&\quad + 2n^{d_1} \exp(2H(\beta^{k_n}) - \beta^{-3k_n/5}/2\theta_1) + 2n^{d_2} \exp(2H(\beta^{k_n}) \\
&\quad - \beta^{-3k_n/5}/2\theta_2) \\
&\leq 4 \sum_{k \geq k_0} \exp(-H(\beta^{k+1})) + 2n^{d_1} \exp(2H(\beta^{k_n}) - \beta^{-3k_n/5}/2\theta_1) \\
&\quad + 2n^{d_2} \exp(2H(\beta^{k_n}) - \beta^{-3k_n/5}/2\theta_2).
\end{aligned}$$

From the assumption $r < 3/5$, we have

$$\sum_{k \geq k_0} \exp(-H(\beta^{k+1})) \leq \sum_{k \geq k_0} \exp(-\beta^{-(k+1)r/2}),$$

which is summable. Same argument shows that the second and the third terms go to zero as $n \to \infty$. This proves the Theorem.

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