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ON PERTURBATION OF ROOTS OF POLYNOMIALS BY NEWTON'S INTERPOLATION FORMULA

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1. Introduction

Much research has been done to estimate the magnitudes of the changes of the roots from the perturbed polynomials. For more information and references on such work, see [2,4,5,10,17].

In 1984 and 1990 Tulovsky14,15] treated this problem by using Newton's interpolation formula and Rouche's Theorem. In our work, the approach used follows closely that of Tulovsky[14,15].

Let $p(z) = (z-q_1)\cdots(z-q_n)$ be a polynomial of degree n with roots $q_1, \dots, q_n$. We denote by $Q_n = \{q_1, \dots, q_n\}$ the set of roots $p(z)$. And let $r(z)$ be a perturbing polynomial of degree $\leq n-1$. Tulovsky[14,15] proposed and solved the following problems:

Let $\tilde{q}_1, \dots, \tilde{q}_n$ be the roots of the perturbed polynomial $p(z) + r(z)$ and let $\rho \geq 0$ be a given number. Then what are necessary and sufficient conditions on $r(z)$ so that

$$|q_i - \tilde{q}_i| \leq \rho, \quad i = 1, 2, \dots, n \text{ ?} \tag{1.1}$$

Tulovsky showed that (see Theorem 1.2) it is possible to give necessary conditions and sufficient conditions analogous to them for (1.1) to be satisfied. For the proof, he introduced for each subset $\beta \subseteq Q_n$, a polynomial $P_\beta(\rho, |q_i - q_j|)$ of variables ρ and $|q_i - q_j|$, (which satisfy the specific properties which are specified in section 3), where $i,\ j = 1, 2, \dots, n$ *and* $i \neq j$. Using the properties (3.2) of $P_\beta(\rho, |q_i - q_j|)$, Tulovsky gave the following result by Newton's interpolation formula:

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THEOREM 1.2. (Tulovsky[14,15]) *Let* $p(z) = (z-q_1)\cdots(z-q_n)$, *degree* $r(z) \leq n-1$, $p(z)+r(z) = (z-\tilde{q}_1)\cdots(z-\tilde{q}_n)$. *For given* $\rho \geq 0$, *if the roots* $\tilde{q}_1,\ldots,\tilde{q}_n$ *of* $p(z)+r(z)$ *can be indexed in such a way that* $|q_i-\tilde{q}_i| \leq \rho$, $i=1,2,\ldots,n$, *then for any nonempty subset* $\beta = \{q_{i_1},\ldots q_{i_m}\} \subseteq Q_n$, *the following estimate holds;*

$$|r[\beta]| \leq P_\beta(\rho, |q_i - q_j|) ,$$

where $r[\beta]$ *is the* $(m-1)$*-th divided difference of* $r(z)$, *calculated at the points* $q_{i_1},\ldots q_{i_m}$.

Conversely, if $|r[\beta]| \leq P_\beta(\rho,|q_i-q_j|)$ *for all nonempty subsets* $\beta \subseteq Q_n$, *then there exists some constant* $C(n)$ *depending only on* n *so that the roots* $\tilde{q}_1,\ldots,\tilde{q}_n$ *of* $p(z)+r(z)$ *can be indexed in such a way that*

$$|q_i-\tilde{q}_i| \leq C(n)\rho, \quad i=1,2,\ldots,n.$$

It turns out that from the polynomials $P_\beta(\rho,|q_i-q_j|)$ given in [14,15], it would be very complicated to find explicitly the constant $C(n)$ which is given in the converse part of Theorem (1.2) and Tulovsky does not give an estimate for $C(n)$. Hence the main purpose of this work is to solve the following problems:

A). Find polynomials $|r[\beta]| \leq P_\beta(\rho,|q_i-q_j|)$ for all nonempty subsets $\beta \subseteq Q_n$ which satisfy the properties given in (3.2), that are in some sense minimal.

B). From the minimal polynomials $P_\beta(\rho,|q_i-q_j|)$, estimate the constant $C(n)$ in the above theorem, and determine asymptotic properties of $C(n)$ grow as $n \to \infty$?

2. Preliminaries

Most of the work in this section is devoted to deriving a new formula for the divided difference $r[\beta]$. Before proceeding a short comment on notation and some known results from the theory of divided differences are needed. The most detailed exposition of their properties can be found in Milne-Thomson[12], The Calculus of Finite Differences. For other references on the divided difference, see [6,8,9].

DEFINITION 2.1. Let $p(z)$ be a polynomial in the complex variable z. The first divided difference of $p(z)$ is denoted by the relation

$$p[z_0, z_1] = \frac{p(z_0) - p(z_1)}{z_0 - z_1}.$$

The n-th divided difference is defined by induction in terms of the (n-1)-th one by the formula

$$p[z_0, \dots, z_n] = \frac{p[z_0, \dots, z_{n-2}, z_n] - p[z_0, \dots, z_{n-2}, z_{n-1}]}{z_n - z_{n-1}}. \tag{2.1}$$

In order to derive a new formula for $r[\beta]$ which is useful in studying perturbation of roots, we need the following lemma.

LEMMA 2.2. [6,12]

$$p[z_0, \dots, z_n] = \frac{1}{2\pi i} \int_\Gamma \frac{p(z)}{(z - z_0) \cdots (z - z_n)} dz,$$

where the points $z_0, \dots, z_n$ *lie inside the contour* Γ.

By Cauchy's integral formula, we have the following estimate;

$$|p[z_0, \dots, z_n]| \le \frac{1}{n!} \sup_{z \in D}(|p^{(n)}(z)|), \tag{2.2}$$

where D is any convex region in the complex plane, containing $z_0, \dots,$ z_n. For n+1 coincident arguments z_0, we obtain the equality

$$p[z_0, \dots, z_n] = \frac{1}{n!} p^{(n)}(z_0). \tag{2.3}$$

If $p(z)$ is a polynomial of degree n, then by Newton's interpolation formula, $p(z)$ can be reconstructed uniquely from the values of the divided differences at $z_0, \dots, z_n$ as follows:

$$p(z) = p[z_0] + p[z_0, z_1](z - z_0) + \cdots + p[z_0, \dots, z_n](z - z_0) \cdots (z - z_{n-1}).$$

For more information and references to these discoveries, see [6,12].

Now we will follow some notations from Tulovsky[15] and [13]. Let $p(z)$ be a polynomial with degree n. we denote the set of roots of $p(z)$ by $Q_n = \{q_1, \dots, q_n\}$, the letters $\alpha, \beta, \gamma, \dots$ will denote subsets of Q_n, and $|\alpha|, |\beta|, |\gamma|, \dots$ the number of elements in these subsets. For $\alpha \subseteq Q_n$ we denote by $p[\alpha]$ the divided difference of $p(z)$, calculated at the points $q_i \in \alpha$. If $\alpha = \emptyset$, then $p[\alpha] = 0$. If $\alpha, \beta, \gamma, \dots$ are subsets of Q_n, then we shall denote by $\alpha', \beta', \gamma', \dots$ complements of these subsets in Q_n. We set for any $\alpha \subseteq Q_n$,

$$(z-q)^\alpha = \prod_{q_i \in \alpha}(z-q_i), \qquad (z-q)^\alpha = 1 \ for \ \alpha = \emptyset,$$

$$(q-\tilde{q})^\alpha = \prod_{q_i \in \alpha}(q_i-\tilde{q}_i), \qquad (q-\tilde{q})^\alpha = 1 \ for \ \alpha = \emptyset.$$

REMARK 2.3. If a polynomial $p(z)$ has multiple roots, then each root must be counted in the set Q_n as many times as its multiplicity, and any subset α of Q_n may contain in this case some copies of this multiple roots, while all other copies of this multiple roots will be contained in the complement α'.

Now for the divided difference of the perturbing polynomial for $r[z]$, we will start to construct a new formula for $r[\beta]$ in terms of $q_i - \tilde{q}_j$. From Lemma 2.2 and (2.3), we obtain the following results.

LEMMA 2.4.

$$\frac{1}{2\pi i}\int_\Gamma \frac{(z-p_1)\cdots(z-p_m)}{(z-q_1)\cdots(z-q_n)}\,dz = \begin{cases} 0 & if\, m < n-1, \\ 1 & if\, m = n-1, \\ \sum_{i=1}^{n}(q_i - p_i) & if\, m = n, \end{cases}$$

where Γ *is a contour containing* $q_1, \dots, q_n$.

From Lemma 2.2 and Lemma 2.4, we get the following result.

LEMMA 2.5. *If* $p(z) = (z-q_1)(z-q_2)(z-q_3)$, $deg\ r(z) \leq 2$ *and* $p(z) + r(z) = (z-\tilde{q}_1)(z-\tilde{q}_2)(z-\tilde{q}_3)$, *then*

$$\begin{aligned} r[q_i] &= (q_i-\tilde{q}_1)(q_i-\tilde{q}_2)(q_i-\tilde{q}_3) \quad for \ \ 1 \leq i \leq 3, \\ r[q_i,q_j] &= (q_i-\tilde{q}_i)(q_i-\tilde{q}_k) + (q_j-\tilde{q}_j)(q_j-\tilde{q}_k) + (q_i-\tilde{q}_i)(q_j-\tilde{q}_j) \\ &\qquad for\ 1 \leq i,j,k \leq 3\ and\ i \neq j \neq k, \\ r[q_1,q_2,q_3] &= (q_1-\tilde{q}_1) + (q_2-\tilde{q}_2) + (q_3-\tilde{q}_3). \end{aligned}$$

REMARK 2.6. Let $Q_n = \{q_1, \dots, q_n\}$ be a fixed set. Then for any subset $\{q_i, q_j, q_k, \dots\} \subseteq Q_n$, we shall always set $i < j < k < \cdots$ throughout our paper.

Now we will define $(q_\alpha - \tilde{q})^\nu$ as follows; for any subset $\alpha \subseteq \beta$ such that $|\beta| = m \leq n$, set $\alpha = \{q_{\alpha_1}, q_{\alpha_2}, \cdots\}$, $\beta' = \{q_{c_1}, q_{c_2}, \cdots\} \subseteq Q_n$. Choose $\nu = \{q_{c_{j_1}}, q_{c_{j_2}}, \dots, q_{c_{j_{|\nu|}}}\} \subseteq \beta'$ so that $|\nu| = n + 1 - m - |\alpha|$, then we will define

$$(q_\alpha - \tilde{q})^\nu = (q_{\alpha_{i_1}} - \tilde{q}_{c_{j_1}})(q_{\alpha_{i_2}} - \tilde{q}_{c_{j_2}}) \cdots (q_{\alpha_{i_{|\nu|}}} - \tilde{q}_{c_{j_{|\nu|}}})$$

so that $i_1 = j_1$, $i_2 = j_2 - 1, \dots, i_{|\nu|} = j_{|\nu|} - |\nu| + 1$. Here we also set for any $\alpha, \nu \subseteq Q_n$,

$$(q_\alpha - \tilde{q})^\nu = \begin{cases} 1 & \text{for } |\nu| = 0 \\ 0 & \text{for } |\nu| < 0. \end{cases}$$

The next result is basic to the results in this paper. For our work we need a formula for $r[\beta]$ in terms of differences $q_i - \tilde{q}_j$.

THEOREM 2.7. *Suppose that* $p(z) = (z - q_1) \cdots (z - q_n)$, *deg* $r(z) \leq n - 1$, $p(z) + r(z) = (z - \tilde{q}_1) \cdots (z - \tilde{q}_n)$ *and* $Q_n = \{q_1, \dots, q_n\}$. *Then for any subset* $\beta \subseteq Q_n$ *such that* $|\beta| = m \leq n$, *we have*

$$r[\beta] = \frac{1}{2\pi i} \int_\Gamma \frac{(z - \tilde{q}_1) \cdots (z - \tilde{q}_n)}{(z - q)^\beta} dz = \sum_{\substack{|\alpha| \geq 1 \\ \alpha \subseteq \beta}} (q - \tilde{q})^\alpha \sum_{\substack{\nu \subseteq \beta' \\ |\nu| = n+1-m-|\alpha|}} (q_\alpha - \tilde{q})^\nu. \tag{2.4}$$

REMARK 2.8. Since $r(z) = (z - \tilde{q}_1) \cdots (z - \tilde{q}_n) - (z - q_1) \cdots (z - q_n)$, the first equality in (2.4) follows trivially from Lemma 2.2. What needs to be established is the second equality.

Proof. (Theorem 2.7.) We are going to prove the theorem by induction on n. If $n = 3$, by Lemma 2.5, (2.4) is certainly solid. Now assume that it is true for $n - 1$ and that $m < n$. Clearly it is enough to show that for any subset $\beta = \{q_1, \ldots, q_m\} \subset Q_n, m < n$, (2.4) is satisfied. From Lemma 2.2, for a contour Γ containing $q_1, \ldots, q_m$, we get

$$\frac{1}{2\pi i}\int_\Gamma \frac{(z-\tilde{q}_1)\cdots(z-\tilde{q}_n)}{(z-q_1)\cdots(z-q_m)}\,dz$$

$$=\frac{1}{2\pi i}\int_\Gamma \frac{(z-\tilde{q}_2)\cdots(z-\tilde{q}_n)}{(z-q_2)\cdots(z-q_m)}\,dz \tag{2.5}$$

$$+(q_1-\tilde{q}_1)\frac{1}{2\pi i}\int_\Gamma \frac{(z-\tilde{q}_2)\cdots(z-\tilde{q}_n)}{(z-q_1)\cdots(z-q_m)}\,dz. \tag{2.6}$$

Take $\bar{Q}_{n-1} = \{q_2, \ldots, q_n\}$, $\bar{\beta} = \{q_2, \ldots, q_m\}$ and apply the induction hypothesis to (2.5) to obtain

$$\frac{1}{2\pi i}\int_\Gamma \frac{(z-\tilde{q}_2)\cdots(z-\tilde{q}_n)}{(z-q_2)\cdots(z-q_m)}\,dz = \sum_{\substack{|\alpha|\geq 1\\ \alpha\subseteq\bar{\beta}}}(q-\tilde{q})^\alpha \sum_{\substack{\nu\subseteq \bar{Q}_{n-1}/\bar{\beta}\\ |\nu|=n+1-m-|\alpha|}}(q_\alpha-\tilde{q})^\nu.$$

For the second part (2.6),

$$\frac{1}{2\pi i}\int_\Gamma \frac{(z-\tilde{q}_2)\cdots(z-\tilde{q}_n)}{(z-q_1)\cdots(z-q_m)}\,dz$$
$$=\frac{1}{2\pi i}\int_\Gamma \frac{(z-\tilde{q}_{m+1})(z-\tilde{q}_2)\cdots(z-\tilde{q}_m)(z-\tilde{q}_{m+2})\cdots(z-\tilde{q}_n)}{(z-q_1)\cdots(z-q_m)}\,dz.$$

Now let us set

$$\begin{cases} \bar{q}_1 = \tilde{q}_{m+1} & \\ \bar{q}_i = \tilde{q}_i & \text{for } 2\leq i\leq m \\ \bar{q}_i = \tilde{q}_{i+1} & \text{for } m+1\leq i\leq n-1 \end{cases} \tag{2.7}$$

Then taking $\bar{Q}_{n-1} = \{q_1, \ldots, q_{n-1}\}$ and applying the induction hypothesis, we obtain

$$\frac{1}{2\pi i}\int_\Gamma \frac{(z-\tilde{q}_{m+1})(z-\tilde{q}_2)\cdots(z-\tilde{q}_m)(z-\tilde{q}_{m+2})\cdots(z-\tilde{q}_n)}{(z-q_1)\cdots(z-q_m)}\,dz$$
$$=\sum_{\substack{|\alpha|\geq 1\\ \alpha\subseteq\beta}}(q-\bar{q})^\alpha \sum_{\substack{\nu\subseteq \bar{Q}_{n-1}/\beta\\ |\nu|=n-m-|\alpha|}}(q_\alpha-\bar{q})^\nu.$$

Now we would like to show that

$$\text{(2.8)}\qquad \sum_{\substack{|\alpha|\geq 1\\ \alpha\subseteq\beta}} (q-\tilde{q})^{\alpha} \sum_{\substack{\nu\subseteq Q_{n-1}/\beta\\ |\nu|=n+1-m-|\alpha|}} (q_{\alpha}-\tilde{q})^{\nu}$$

$$\text{(2.9)}\qquad +(q_1-\tilde{q}_1)\sum_{\substack{|\alpha|\geq 1\\ \alpha\subseteq\beta}} (q-\tilde{q})^{\alpha} \sum_{\substack{\nu\subseteq Q_{n-1}/\beta\\ |\nu|=n-m-|\alpha|}} (q_{\alpha}-\tilde{q})^{\nu}$$

(2.10)

$$=\sum_{\substack{|\gamma|\geq 1\\ \gamma\subseteq\beta}} (q-\tilde{q})^{\gamma} \sum_{\substack{\omega\subseteq\beta'\\ |\omega|=n+1-m-|\gamma|}} (q_{\gamma}-\tilde{q})^{\omega}.$$

In order to show the above equality, it is sufficient to prove the following;

(1) The number of terms of (2.8) and (2.9) is equal to the number of terms of (2.10).

(2) Every terms of (2.8) and (2.9) is a term of (2.10). (Here a term means an expression of the form $(q_{i_1}-\tilde{q}_{j_1})\cdots(q_{i_m}-\tilde{q}_{j_m})$.)

From the definitions of $(q-\tilde{q})^{\alpha}$ and $(q_{\alpha}-\tilde{q})^{\nu}$, the number of (2.8) and (2.9) is

$$\binom{n-1}{n-m+1}+\binom{n-1}{n-m}=\binom{n}{n-m+1}.$$

Since the right side is the number of terms of (2.10), (1) is proved.

For the second statement (2), it is easy to see that every term of (2.8) is a term of (2.10) because if $\alpha\subseteq\bar{\beta}$ and $\nu\subseteq\bar{Q}_{n-1}/\bar{\beta}$ are given in (2.8), then we can see that $\gamma=\alpha$ and $\omega=\nu$ in (2.10). Next we have to show that every term of (2.9) is a term of (2.10). Here we need to consider two cases:

$$q_1\notin\alpha=\{q_{\alpha_1},q_{\alpha_2},\dots\}\subseteq\beta,\quad q_1\in\alpha=\{q_{\alpha_1},q_{\alpha_2},\dots\}\subseteq\beta.$$

In either case, we can also see that every term of(2.9) is a term of (2.10) by using (2.7).

For the detail proof of (2), see [13].

As an immediate consequence of this theorem we obtain the following results, which can also be obtained classically, eg., Milne-Thomson [12].

COROLLARY 2.9. *If all $\tilde{q}_i = 0$ in Theorem 2.7, then we have*

$$\frac{1}{2\pi i}\int_\Gamma \frac{z^n}{(z-q_1)\cdots(z-q_m)}\,dz = \sum_{\substack{i_1+i_2+\cdots+i_m=n+1-m \\ i_1,i_2,\ldots,i_m\in Z^+\cup\{0\}}} q_1^{i_1}q_2^{i_2}\cdots q_m^{i_m},$$

which is a homogenous polynomial with degree $n+1-m$ with respect to $q_1,\ldots,q_m$, where $q_1,\ldots,q_m$ lie inside the contour Γ.

(Here note that from Remark 2.4, $q_1,\ldots,q_m$ need not be necessarily distinct.)

COROLLARY 2.10. *If we want to expand $p(z)=(z-b_1)\cdots(z-b_n)$ by $(z-b_1)\cdots(z-b_n)=c_n(z-z_0)^n+c_{n-1}(z-z_0)^{n-1}+\cdots+c_1(z-z_0)+c_0$, then ,by Theorem 2.7, the coefficients c_m can be expressed as follows;*

$$c_m = \sum_{b_{i_k}\in\{b_1,\ldots,b_n\}} (z_0-b_{i_1})\cdots(z_0-b_{i_{n-m}}) \quad \textit{for} \quad m=0,1,\ldots,n.$$

Let us write down the formula for $r[\beta]$ for the simple case $n=4$.

EXAMPLE 2.11. For $n=4$, we have the following forms;

$$r[q_i] = (q_i-\tilde{q}_1)(q_i-\tilde{q}_2)(q_i-\tilde{q}_3)(q_i-\tilde{q}_4) \qquad \text{for } 1\le i\le 4,$$

$$\begin{aligned} r[q_i,q_j] =& (q_i-\tilde{q}_i)(q_i-\tilde{q}_k)(q_i-\tilde{q}_m)+(q_j-\tilde{q}_j)(q_j-\tilde{q}_k)(q_j-\tilde{q}_m) \\ &+(q_i-\tilde{q}_i)(q_j-\tilde{q}_j)(q_i-\tilde{q}_k) \\ &\text{for } 1\le i\ne j\ne k\ne m\le 4, \end{aligned}$$

$$\begin{aligned} r[q_i,q_j,q_k] =& (q_i-\tilde{q}_i)(q_i-\tilde{q}_m)+(q_j-\tilde{q}_j)(q_j-\tilde{q}_m) \\ &+(q_k-\tilde{q}_k)(q_k-\tilde{q}_m)+(q_i-\tilde{q}_i)(q_j-\tilde{q}_j) \\ &+(q_i-\tilde{q}_i)(q_k-\tilde{q}_k)+(q_j-\tilde{q}_j)(q_k-\tilde{q}_k) \\ &\text{for } 1\le i\ne j\ne k\ne m\le 4, \end{aligned}$$

$$r[q_1,q_2,q_3,q_4] = (q_1-\tilde{q}_1)+(q_2-\tilde{q}_2)+(q_3-\tilde{q}_3)+(q_4-\tilde{q}_4),$$

where $p(z)=(z-q_1)(z-q_2)(z-q_3)(z-q_4)$, $\deg r(z)\le 3$.

3. Construction of the Polynomials $P_\beta(\rho, |q_i - q_j|)$

In this section we will define polynomials $P_\beta(\rho, |q_i - q_j|)$ from the formula obtained in the previous section and will show that the constructed polynomials $P_\beta(\rho, |q_i - q_j|)$ are minimal in some sense.

If $|q_i - \tilde{q}_i| \le \rho$ for all $i = 1, \dots, n$, then we get the estimate

$$\begin{aligned}|(q_\alpha - \tilde{q})^\nu| &= |q_{\alpha_{i_1}} - \tilde{q}_{c_{j_1}}|\,|q_{\alpha_{i_2}} - \tilde{q}_{c_{j_2}}| \cdots |q_{\alpha_{i_{|\nu|}}} - \tilde{q}_{c_{j_{|\nu|}}}| \\ &= |q_{\alpha_{i_1}} - q_{c_{j_1}} + q_{c_{j_1}} - \tilde{q}_{c_{j_1}}| \cdots |q_{\alpha_{i_{|\nu|}}} - q_{c_{j_{|\nu|}}} + q_{c_{j_{|\nu|}}} - \tilde{q}_{c_{j_{|\nu|}}}| \\ &\le (|q_{\alpha_{i_1}} - q_{c_{j_1}}| + \rho) \cdots (|q_{\alpha_{i_{|\nu|}}} - q_{c_{j_{|\nu|}}}| + \rho).\end{aligned}$$

Motivated by the above estimate, we shall define the polynomials $P_\beta(\rho, |q_i - q_j|)$ as follows:

DEFINITION 3.1. Let $Q_n = \{q_1, \dots, q_n\}$, then for any subset $\beta \subseteq Q_n$ and $\rho \ge 0$, we will define polynomials P_β with respect to variables ρ and $|q_i - q_j|$ by the formula

$$P_\beta(\rho, |q_i - q_j|) = \sum_{\substack{|\alpha| \ge 1 \\ \alpha \subseteq \beta}} \rho^{|\alpha|} \sum_{\substack{\nu \subseteq \beta' \\ |\nu| = n+1-m-|\alpha|}} (q_\alpha - q)^\nu_\rho,$$

where $(q_\alpha - q)^\nu_\rho = (|q_{\alpha_{i_1}} - q_{c_{j_1}} + \rho) \cdots (|q_{\alpha_{i_{|\rho|}}} - q_{c_{j_{|\rho|}}} + \rho)$.

Let's look at the polynomials that correspond to Example 2.11.

EXAMPLE 3.2. If $p(z) = (z - q_1)(z - q_2)(z - q_3)(z - q_4)$, then we have

$$P_{\{q_1\}}(\rho, |q_i - q_j|) = \rho(|q_1 - q_2| + \rho)(|q_1 - q_3| + \rho)(|q_1 - q_4| + \rho),$$

$$\begin{aligned}P_{\{q_1, q_2\}}(\rho, |q_i - q_j|) =& \rho(|q_1 - q_3| + \rho)(|q_1 - q_4| + \rho) \\ &+ \rho(|q_2 - q_3| + \rho)(|q_2 - q_4| + \rho) \\ &+ \rho^2(|q_1 - q_3| + \rho) + \rho^2(|q_2 - q_4| + \rho),\end{aligned}$$

$$\begin{aligned}P_{\{q_1, q_2, q_3\}}(\rho, |q_i - q_j|) =& \rho(|q_1 - q_4| + \rho) + \rho(|q_2 - q_4| + \rho) \\ &+ \rho(|q_3 - q_4| + \rho) + 3\rho^2,\end{aligned}$$

$$P_{\{q_1, q_2, q_3, q_4\}}(\rho, |q_i - q_j|) = 4\rho.$$

From the definition of P_β we can see that

(3.1);

(1) $P_{Q_n}(\rho, |q_i - q_j|) = n\rho$.

(2) P_β is a universal polynomial of ρ and $|q_i - q_j|$ that depends only on the subset β and is dependent of the perturbing polynomial $r(z)$.

(3) If the roots $\tilde{q}_1, \cdots, \tilde{q}_n$ of $p(z) + r(z)$ satisfy $|q_i - \tilde{q}_i| \leq \rho$, $1 \leq i \leq n$, then $|r[\beta]| \leq P_\beta(\rho, |q_i - q_j|)$.

We can also check that the polynomials $P_\beta(\rho, |q_i - q_j|)$ satisfy the following properties 1) - 5) of Tulovsky ;

(3.2) ;

(1) $P_\beta(\rho, |q_i - q_j|)$ depends only on the variables ρ and $|q_i - q_j|$, where $q_j \in \beta'$, $q_i \in \beta$.

(2) $P_\beta(\rho, |q_i - q_j|)$ is a *homogeneous* polynomial of degree $n + 1 - |\beta|$, jointly in ρ and $|q_i - q_j|$.

(3) $P_\beta(\rho, |q_i - q_j|)$ has total degree at most $n - |\beta|$ with respect to $|q_i - q_j|$.

(4) $P_\beta(\rho, |q_i - q_j|)$ has degree at most one in each variable $|q_i - q_j|$.

(5) If $q_j \in \beta'$, then every term in $P_\beta(\rho, |q_i - q_j|)$ contains at most one factor of the form $|q_i - q_j|$ with $q_i \in \beta$. (A term means an expression of the form

$$\rho^k |q_{i_1} - q_{j_1}||q_{i_2} - q_{j_2}| \cdots |q_{i_m} - q_{j_m}| \ .)$$

Moreover from the properties 2), 3) of (3.2) and the new formula for $r[\beta]$, it is not hard to see that the following properties are satisfied.

(3.3);

for $\beta \subseteq Q_n$ such that $|\beta| = m$,

(1) $r[\beta]$ has $\binom{n}{n+1-m}$ terms.

(2) For a fixed $k \leq m$, in $P_\beta(\rho, |q_i - q_j|)$ the number of terms containing ρ^k is $\binom{m}{k}$ and for each $|\alpha| = k$ the number of terms of the form $(q_\alpha - q)^\nu_\rho$ is $\binom{n-m}{n+1-m-k}$.

The above properties make it possible to accurately estimate the constant $C(n)$ in Theorem 1.2 and will also be useful in obtaining other results.

Next we will give an example so that

$$r[\beta] = P_\beta(\rho, |q_i - q_j|) \ .$$

EXAMPLE 3.3. Let $Q_n = \{q_1, \ldots, q_n\}$ be a decreasing sequence on the real line $\mathbb{R}$ and $\tilde{q}_i = q_i - \rho$, $i = 1, \ldots, n$. Then for $\beta = \{q_1, \ldots, q_m\} \subseteq Q_n$, we can see that $(q - \tilde{q})^\alpha = \rho^{|\alpha|}$ for any $\alpha \subseteq \beta$ and

$$\begin{aligned} |(q_\alpha - \tilde{q})^\nu| &= |q_{\alpha_{i_1}} - \tilde{q}_{c_{j_1}}||q_{\alpha_{i_2}} - \tilde{q}_{c_{j_2}}| \cdots |q_{\alpha_{i_{|\nu|}}} - \tilde{q}_{c_{j_{|\nu|}}}| \\ &= (|q_{\alpha_{i_1}} - q_{c_{j_1}}| + \rho) \cdots (|q_{\alpha_{i_{|\nu|}}} - q_{c_{j_{|\nu|}}}| + \rho), \end{aligned}$$

where $\nu \subseteq \beta'$. We therefore obtain the equality $r[\beta] = P_\beta(\rho, |q_i - q_j|)$ when $\beta = \{q_1, \ldots, q_m\} \subseteq Q_n$.

DEFINITION 3.4. Let $Q_n = \{q_1, \ldots, q_n\}$ and $\deg r(z) \leq n - 1$. Let Ξ_n be the collection of all polynomials $\{P_\beta\}$ satisfying the following conditions;

i) If the roots $\tilde{q}_1, \cdots, \tilde{q}_n$ of $p(z) + r(z)$ satisfy $|q_i - \tilde{q}_i| \leq \rho$, then $|r[\beta]| \leq P_\beta(\rho, |q_i - q_j|)$ for all non-empty subsets $\beta \subseteq Q_n$.
ii) P_β satisfies the 5-properties of (3.2).

Now we will define a relation $\leq$ on the class Ξ_n by: $\{P_\beta\} \leq \{P'_\beta\}$ if $P_\beta(\rho, |q_i - q_j|) \leq P'_\beta(\rho, |q_i - q_j|)$ for all non-empty subsets $\beta \subseteq Q_n$, $\rho \geq 0$ and all $|q_i - q_j|$. Then we can easily check that $(\Xi_n, \leq)$ is a partially ordered set.

REMARK 3.5. For the universal polynomials $\{\tilde{P}_\beta\}$ and $\{\overline{P}_\beta\}$ defined by Tulovsky[14,15] in 1984 and 1990 respectively, it turns out that $\{\tilde{P}_\beta\} = \{P_\beta\}$ for all $n \geq 3$, $\{\tilde{P}_\beta\} \geq \{P_\beta\}$ for all $n > 3$ and $\{\overline{P}_\beta\} \geq \{P_\beta\}$ for all n.

THEOREM 3.6. *The class $\{\tilde{P}_\beta\}$ consisting of the polynomials $P_\beta(\rho, |q_i - q_j|)$ of Definition 3.1 is a minimal element in Ξ_n for all n.*

Proof. Suppose that $\{P'_\beta\} \leq \{P_\beta\}$ on Ξ_n, $n \geq 1$. Then by definition, $P'_\beta(\rho, |q_i - q_j|) \leq P_\beta(\rho, |q_i - q_j|)$ for all non-empty subsets $\beta \subseteq Q_n$, $\rho \geq 0$ and all Q_n. From the property (2) of (3.2) and using the fact that $\rho, \{q_1 \cdots q_n\}$ are arbitrary, we can see that every term in P'_β is a term in P_β if the relation $P'_\beta \leq P_\beta$ is to hold. Now for $\beta = \{q_{i_1}, q_{i_2}, \ldots, q_{i_m}\}$, we would like to choose $Q_n = \{q_1, \ldots, q_n\}$ so that $|r[\beta]| = P_\beta(\rho, |q_i - q_j|)$ as follows; in Example 3.3, switch q_j to q_{i_j}, $1 \leq j \leq m$, and set $q_{m+j} = q_{c_j}$ for $j \geq 1$, where $q_{c_j} \in \beta' = \{q_{c_1}, q_{c_2}, \ldots, \}$. Here note that $q_{i_1} \geq q_{i_2} \geq$

$\cdots q_{i_m} \geq q_{c_1} \geq q_{c_2} \geq \cdots$ and $\tilde{q}_i = q_i - \rho,\ i = 1, \ldots, n$. Hence, for the $Q_n = \{q_1, \cdots, q_n\}$ obtained by rearranging $\{q_1, \cdots, q_n\}$ in Example 3.3, we can see that for $\beta = \{q_{i_1}, q_{i_2}, \cdots q_{i_m}\}$, $|r[\beta]| = P_\beta(\rho, |q_i - q_j|)$ by the condition (1) of (3.2). Since $|r[\beta]| \leq P'_\beta(\rho, |q_i - q_j|)$, we obtain $P_\beta(\rho, |q_i - q_j|) = P'_\beta(\rho, |q_i - q_j|)$. That is, $\{P_\beta\}$ is a minimal element in Ξ_n for $n \geq 1$.

4. Tulovsky's theorem and estimation or the universal constant $C(n)$

In this section, we will obtain an explicit relation in terms of polynomials $P_\beta(\rho, |q_i - q_j|)$ between the perturbations of roots and a perturbations of coefficients by using Newton's interpolation formula.

We start by introducing some notations. The set of all complex numbers is denoted by $\mathbb{C}$. By $B(z_0, \rho)$, we shall always mean the closed disk of radius ρ centered at z_0. If S is any bounded set in $\mathbb{C}$, its diameter is given by

$$dia(S) = \sup_{z, z' \in S} (|z - z'|) .$$

LEMMA 4.1. (Descarte's rule of signs [7].) *Let $p(z) = b_n z^n + \cdots + b_1 z + b_0$ be a real polynomial (not the zero polynomial) and let v denote the number of sign changes in the sequence $\{b_k\}$ of its non-zero coefficients, and let r denote the number of its real positive roots (each root counted with its proper multiplicity), then $v - r$ is even and non-negative.*

Now we are going to prove (B).

THEOREM 4.2. *Let $p(z) = (z - q_1) \cdots (z - q_n)$, $r(z)$ be a polynomial with degree $\leq n - 1$ and $\tilde{p}(z) = p(z) + r(z) = (z - \tilde{q}_1) \cdots (z - \tilde{q}_n)$.*

For a given $\rho \geq 0$, if $|q_i - \tilde{q}_i| \leq \rho$ for all i, then for non-empty subset $\beta \subseteq Q_n$,

$$|r[\beta]| \leq P_\beta(\rho, |q_i - q_j|) \quad . \tag{4.1}$$

Conversely if (4.1) holds for all non-empty subsets $\beta \subseteq Q_n$, then

1) There exists a constant $\Re(n)$ depending only on n so that the roots $\tilde{q}_1, \cdots, \tilde{q}_n$ of $p(z) + r(z)$ can be indexed in such a way that

$$|q_i - \tilde{q}_i| \leq dia(C_i) - \Re(n)\rho, \ i = 1, \ldots, n, \tag{4.2}$$

where C_i is the connected component of $G = \bigcup_{i=1}^{n} B(q_i, \Re(n)\rho)$ containing q_i, $\Re(n)$ is the positive solution of the equation;

$$k^n - \sum_{m=0}^{n-1} k^m \sum_{t=0}^{n-m-1} \binom{m+1}{n-m-t}\binom{n-m-1}{t}(1+4k)^t = 0\ .$$

2) $\frac{4^{n+1}-10}{9} < \Re(n) < \frac{4^{n+1}-4}{9}$, *i.e.* $\Re(n) \approx \frac{4^{n+1}}{9}$.

Proof. Suppose that $|q_i - \tilde{q}_i| \le \rho$ for all i. From(3.1), for all non-empty subsets $\beta \subseteq Q_n$, $|r[\beta]| \le P_\beta(\rho, |q_i - q_j|)$.

Now we are going to prove the converse part. Since we have shown that our polynomials $P_\beta(\rho, |q_i - q_j|)$ also satisfy the 5-properties in (3.2), we will give roughly the proof, and then estimate the constant $\Re(n)$ and the asymptotic estimate of $\Re(n)$ from the properties (3.3). Now suppose that (4.1) holds. In order to find $\Re(n)$ the proof is based on Rouche's Theorem and Newton's interpolation formula. Let G be the union of all disks $B(q_i, k\rho)$ with boundary Γ. Without loss of generality we assume that $z \in \Gamma$ with $|z - q_1| = k\rho$ and that $|q_1 - q_2| \le |q_1 - q_3| \le \cdots \le |q_1 - q_n|$. By Newton's interpolation formula and the hypothesis (4.1), we have

$$\begin{aligned}
&|r(z)| \\
\le&|r[q_1]| + |r[q_1, q_2]||z - q_1| + \cdots + |r[q_1, \cdots, q_n]||z - q_1| \cdots |z - q_{n-1}| \\
\le&P_{\{q_1\}}(\rho, |q_i - q_j|) + P_{\{q_1, q_2\}}(\rho, |q_i - q_j|)|z - q_1| + \cdots \\
&+ P_{\{q_1, \cdots, q_n\}}(\rho, |q_i - q_j|)|z - q_1| \cdots |z - q_{n-1}|.
\end{aligned}$$

From the estimate
$|q_i - q_j| \le |q_1 - q_i| + |q_1 - q_j| \le 2|q_i - q_j|2|q_1 - z| + |z - q_j|) = 2(k\rho + |z - q_j|)$
for $i < j$, and by using the properties 1), 4) and 5) in (3.2),

$$P_{\{q_1, \cdots, q_s\}}(\rho, 2(k\rho + |z - q_j|))|z - q_1| \cdots |z - q_{s-1}|$$

will have degree at most 1 with respect to each variable $|z - q_\nu|$ for every ν. So now note that

$$\frac{\sum_{s=1}^{n} P_{\{q_1, \cdots, q_s\}}(\rho, 2(k\rho + |z - q_j|))|z - q_1| \cdots |z - q_{s-1}|}{|z - q_1| \cdots |z - q_n|}$$

is a decreasing function of every $|z-q_\nu| > 0$. Hence by replacing $|z-q_\nu|$ by $k\rho$ for every ν, we obtain the estimate

$$\frac{|r(z)|}{|p(z)|} \leq \frac{\sum_{s=1}^{n} P_{\{q_1,\cdots,q_s\}}(\rho, 4k\rho)(k\rho)^{s-1}}{(k\rho)^n} = \frac{\sum_{s=1}^{n} P_{\{q_1,\cdots,q_s\}}(1, 4k)k^{s-1}}{k^n} .$$

From the property (2) in (3.2), $\sum_{s=1}^{n} P_{\{q_1,\cdots,q_s\}}(1, 4k)k^{s-1}$ has degree $\leq n-1$. So, by Lemma 4.1 there is only one positive solution $k = \Re(n)$ of the equation

$$k^n - \sum_{s=1}^{n} P_{\{q_1,\cdots,q_s\}}(1, 4k)k^{s-1} = 0 .$$

For $G = \bigcup_{i=1}^{n} B(q_i, \Re(n)\rho)$, we can see that $\tilde{p}(z) = p(z) + r(z)$ has all it's roots in the region G.

Now Rouche's Theorem gives that in each connected component C_i of G, $p(z)$ and $p(z) + r(z)$ have the same number of roots. Thus we can index the roots $\tilde{q}_1, \cdots, \tilde{q}_n$ of $p(z) + r(z)$ so that

$$|q_i - \tilde{q}_i| \leq dia(C_i) - \Re(n)\rho, \ i = 1, \ldots, n.$$

We will now estimate the constant $\Re(n)$ and the asymptotic estimate of $\Re(n)$.

Using the properties (3.3) we can see that

$$\begin{aligned} &P_{\{q_1,\ldots,q_{m+1}\}}(1, 4k) \\ =&k^n - \sum_{m=0}^{n-1} k^m \sum_{t\geq 0}^{n-m-1} \binom{m+1}{n-m-t}\binom{n-m-1}{t}(1+4k)^t. \end{aligned}$$

Hence the positive solution $Re(n)$ can be found from the equation

$$(4.3)\quad k^n - \sum_{m=0}^{n-1} k^m \sum_{t\leq 0}^{n-m-1} \binom{m+1}{n-m-t}\binom{n-m-1}{t}(1+4k)^t = 0.$$

Let us express the left-hand side of (4.3) as follows;

$$(4.4)\quad k^n - \sum_{m=0}^{n-1} k^m \sum_{t=0}^{n-m-1} \binom{m+1}{n-m-t}\binom{n-m-1}{t}\sum_{i=0}^{t}\binom{t}{i}(4k)^i$$

$$(4.5)\qquad = k^n - b_{n-1}k^{n-1} - b_{n-2}k^{n-2} - \cdots - b_1 k - b_0 .$$

Then we can see that from (4.4),

$$\begin{cases} b_{n-1} = \dfrac{1}{9}(4^{n+1} - 3n - 4) \\ b_{n-2} = \dfrac{1}{54}\{(2n-4)4^{n+1} + 3n^2 + 13n + 16\}. \end{cases}$$

By using the Quadratic formula from the equation from the equation $k^n - b_{n-1}k^{n-1} - b_{n-2}k^{n-2} = 0$, we have $\frac{4^{n+1}-10}{9} < \Re(n)$. It turns out that $b_{n-3} < n^3 4^{n-3}$ and $b_{n-j} < n^{j+1}4^{n-j}$ for $j \geq 4$.

So we get

$$\left(\frac{4^{n+1}-4}{9}\right)^n$$
$$-\left\{b_{n-1}\left(\frac{4^{n+1}-4}{9}\right)^{n-1} + b_{n-2}\left(\frac{4^{n+1}-4}{9}\right)^{n-2} + \cdots + b_1\left(\frac{4^{n+1}-4}{9}\right) + b_0\right\}$$

for $n \geq 1$. Therefore, we obtain

$$\frac{4^{n+1}-10}{9} < \Re(n) < \frac{4^{n+1}-4}{9}.$$

REMARK 4.3. If we don't know the magnitude of the perturbation ρ that $r(z)$ introduces into roots $q_1, \ldots, q_n$ of (4.1). If we choose the smallest ρ so that all the inequalities (4.1) are satisfied, then we know that from (4.2)

$$|q_i - \tilde{q}_i| \leq dia(C_i) - \Re(n)\rho, \quad i = 1, \ldots, n,$$

where C_i is the connected component containing q_i of $G = \cup_{i=1}^{n} B(q_i, \Re(n)\rho)$. And at least one root is perturbed by the amount ρ, i.e., $|q_i - \tilde{q}_i| \geq \rho$ for some i.

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