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L^2–TRANSVERSE FIELDS PRESERVING THE TRANSVERSE RICCI FIELD OF A FOLIATION

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1. Introduction

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional connected Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a complete bundle-like metric g_M with respect to $\mathcal{F}$. Let Ric_D be the transverse Ricci field of $\mathcal{F}$ with respect to the transverse Riemannian connection D which is a torsion-free and g_Q-metrical connection on the normal bundle Q of $\mathcal{F}$. We consider transverse conformal (or, projective) fields of $\mathcal{F}$. It is clear that a transverse Killing field s of $\mathcal{F}$ preserves the transverse Ricci field of $\mathcal{F}$, that is, $\Theta(s)Ric_D = 0$, where $\Theta(s)$ denotes the transverse Lie differentiation with respect to s. The purpose of the present paper is to prove the following theorems:

THEOREM A. *Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional connected Riemannian manifold with a harmonic foliation $\mathcal{F}$ of codimension $q \geq 2$ and a complete bundle-like metric g_M with respect to $\mathcal{F}$. If an L^2-transverse conformal field s of $\mathcal{F}$ with L^2-characteristic function satisfies $\Theta(s)Ric_D = 0$, then s is a transverse Killing field of $\mathcal{F}$.*

THEOREM B. *Let $(M, g_M, \mathcal{F})$ be as Theorem A. If an L^2-transverse projective field s of $\mathcal{F}$ with an L^2-characteristic form ϕ_s satisfies $\Theta(s)$ $Ric_D = 0$, then s is a transverse Killing field of $\mathcal{F}$.*

For compact M without boundary those were proved in [9]. The classical result of Ishihara [2] corresponds to the case of the point foliation on a compact Riemannian manifold without boundary : Let (M, g_M) be a connected, orientable and compact Riemannian manifold without boundary, and let Ric be the Ricci tensor field on M. If a conformal (or, projective) vector field Y on M satisfies $\Theta(Y)Ric = 0$,

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then Y is a Killing vector field on M, where $\Theta(Y)$ denotes the Lie differentiation with respect to Y.

We shall be in C^∞-category and use the following convention on the range of indices :$1 \le i, j, \cdots \le p, p+1 \le \alpha, \beta, \cdots \le p+q$. The Einstein summation convention will be used with respect to those systems of indices.

2. Preliminaries

Let $(M, g_M, \mathcal{F})$ be $(p+q)$-dimensional connected foliated Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a complete bundle-like metric g_M in the sense of Reinhart [12]. The foliation $\mathcal{F}$ induces an integrable subbundle E of the tangent bundle TM over M. The quotient bundle $Q = TM/E$ is called the normal bundle of $\mathcal{F}$. We denote by $\pi : TM \to Q$ the natural projection. The metric g_M defines a map $\sigma : Q \to TM$ with $\pi \circ \sigma = identity$ and induces a metric g_Q in Q ([3], [12]). We denote by D the transverse Riemannian connection in Q which is torsion free and metrical with respect to g_Q ([3], [6], [12]).

In a flat chart $U(x^i, x^\alpha)$ with respect to $\mathcal{F}$ ([12]), a local frame field $\{X_i, X_\alpha\} = \{\partial/\partial x^i, \partial/x^\alpha - A^j_\alpha \partial/\partial x^j\}$ is called the basic adapted frame field of $\mathcal{F}$ ([12], [15]). Here A^j_α are functions on U with $g_M(X_i, X_\alpha) = 0$. It is trivial that X_i (resp. X_α spans $\Gamma(E|_U)$ (resp. $\Gamma(E^\perp|_U)$), where $E^\perp = \sigma(Q)$ denotes the orthogonal complement bundle of E in TM. Hereafter, we omit the term "$|_U$" for simplicity. We set that $g_{ij} = g(X_i, X_j)$, $g_{\alpha\beta} = g(X_\alpha, X_\beta)$, $(g^{ij}) = (g_{ij})^{-1}$, and $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$.

Then we have

LEMMA 1. ([15]) *It holds that* $D_{X_i}\pi(X_\alpha) = 0$, $D_{X_i}D_{X_\alpha}\pi(X_\beta) = 0$ *and* $D_{X_\alpha}\pi(X_\beta) = D_{X_\beta}\pi(X_\alpha)$.

LEMMA 2. ([15]) *It holds that* $[X_\alpha, X_\beta] \in \Gamma(E)$.

The curvature R_D of D is defined by $R_D(X,Y)t = D_X D_Y t - D_Y D_X t - D_{[X,Y]}t$ for any $X, Y \in \Gamma(TM)$ and $t \in \Gamma(Q)$. We notice that $i(X)R_D = 0$, where $i(X)$ denotes the interior product with respect to $X \in \Gamma(E)$. Thus, for any $u, v \in \Gamma(Q)$, the operator $R_D(u,v) : \Gamma(Q) \to \Gamma(Q)$ is a well-defined endomorphism ([3]), that is, $R_D(u,v)t = D_{\sigma(u)}D_{\sigma(v)}t - D_{\sigma(v)}D_{\sigma(u)}t - D_{[\sigma(u),\sigma(v)]}t$. The Ricci operator ρ_D of $\mathcal{F}$

is given by $\rho_D(t) = g^{\alpha\beta} R_D(t, \pi(X_\alpha))\pi(X_\beta)$ ([3]), and the transverse Ricci field Ric_D of $\mathcal{F}$ is defined by

$$Ric_D(t, u) = g_Q(\rho_D(t), u)$$

for any $t, u \in \Gamma(Q)$.

We set

$$V(\mathcal{F}) = \{Y \in \Gamma(TM) | [X, Y] \in \Gamma(E) \ for \ any \ X \in \Gamma(E)\}.$$

An element of $V(\mathcal{F})$ is called an infinitesimal automorphism of $\mathcal{F}$ ([3]). We set

$$\overline{V}(\mathcal{F}) = \{s \in \Gamma(Q) | s = \pi(Y) \ and \ Y \in V(\mathcal{F})\}.$$

The transverse Lie differentiation $\Theta(s)$ with respect to $s = \pi(Y) \in \overline{V}(\mathcal{F})$ is given by

$$\Theta(s)t = \pi([Y, Y_t])$$

for any $t \in \Gamma(Q)$ with $\pi(Y_t) = t$ and $Y_t \in \Gamma(TM)$ ([3],[6]).

LEMMA 3. ([11]) *$(\Theta(s)D)_{[X_\alpha, X_\beta]}\pi(X_\gamma) = 0$ for any $s \in \overline{V}(\mathcal{F})$.*

The transverse divergence $div_D t$ of $t \in \Gamma(Q)$ with respect to D is given by

$$div_D t = g^{\alpha\beta} g_Q(D_{X_\alpha} t, \pi(X_\beta))$$

and the transverse gradient $grad_D f$ of a function f is given by

$$grad_D f = g^{\alpha\beta} X_\alpha(f)\pi(X_\beta)$$

([3], [9]).

For any $s = \pi(Y) \in \overline{V}(\mathcal{F})$, we have an operator $A_D(s) : \Gamma(Q) \to \Gamma(Q)$ defined by

$$A_D(s) = \Theta(s) - D_Y$$

([3]).

DEFINITION. ([3], [7], [15]) If $s \in \overline{V}(\mathcal{F})$ satisfies $\Theta(s)g_Q = 2f_s g_Q$, where f_s is a function on M, then s is called a transverse conformal field (t. c. f.) of $\mathcal{F}$ and f_s the characteristic function of s. If $s \in \overline{V}(\mathcal{F})$

satisfies $\Theta(s)g_Q = 0$, then s is called a transverse Killing field (t. K. f.) of $\mathcal{F}$. If $s \in \overline{V}(\mathcal{F})$ satisfies

$$(\Theta(s)D)_X t = \phi_s(X)t + \phi_s(\sigma(t))\pi(X)$$

for any $X \in \Gamma(TM)$ and $t \in \Gamma(Q)$, where ϕ_s is a 1-form on M, then s is called a transverse projective field (t. p. f.) of $\mathcal{F}$ and ϕ_s the characteristic form of s. If $s \in \overline{V}(\mathcal{F})$ satisfies $\Theta(s)D = 0$, then s is called a transverse affine field (t. a. f.) of $\mathcal{F}$.

PROPOSITION 1. ([9]) *If s is a t.c.f. of $\mathcal{F}$ with characteristic function f_s, then $div_D s$ is a foliated function on M (i.e., $dvi_D s$ has constant values on leaves) and $div_D s = q f_s$.*

PROPOSITION 2. ([9]) *If s is a t.p.f. of $\mathcal{F}$ with characteristic form ϕ_s, then $div_D s$ is a foliated function on M, $d(div_D s) = (q+1)\phi_s$ and $\phi_s(X) = 0$ for any $X \in \Gamma(E)$.*

PROPOSITION 3. ([9]) *If s is a t.c.f. of $\mathcal{F}$ with characteristic function f_s, then it holds that*

$$(\Theta(s)D)_{X_\alpha}\pi(X_\beta) = \{X_\alpha(f_s)\delta^\varepsilon_\beta + X_\beta(f_s)\delta^\varepsilon_\alpha - X_\gamma(f_s)g_{\alpha\beta}g^{\gamma\varepsilon}\}\pi(X_\varepsilon)$$

where δ^α_β denotes the Kronecker delta.

Let ∇^M be the Levi-Civita connection with respect to g_M. Then an operator $\Delta_D : \Gamma(Q) \to \Gamma(Q)$ is defined by

$$\Delta_D s = -g^{\alpha\beta}(D_{X_\alpha}D_{X_\beta}s - D_{\nabla^M_{X_\alpha}X_\beta}s) - g^{ij}(D_{X_i}D_{X_j}s - D_{\nabla^M_{X_i}X_j}s)$$

for any $s \in \Gamma(Q)$ ([5], [10]).

PROPOSITION 4. ([9]) *Let $s \in \overline{V}(\mathcal{F})$. If s is a t.a.f. of $\mathcal{F}$,then $\Delta_D(s) = D_{\sigma(\tau)}s + \rho_D(s)$ and $d(div_D s) = 0$, where τ is the tension field defined by $\tau = -g^{ij}(D_{X_i}\pi)(X_j)$.*

PROPOSITION 5. ([5], [13]) *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$, and let Δ be the Laplace-Beltrami operator acting on functions on M. Then it holds the following decomposition of Δ :*

$$\Delta f = \Box' f + \Box''_0 f + Hf \qquad \textit{for any function } f \textit{ on } M.$$

In Proposition 5, an operator $\Box'$ is defined by $\Box' f = -g^{ij} X_i(X_j f) + g^{ij}(\nabla^M_{X_i} X_j)_E f$, where $(\ \ )_E$ denotes the E-component of ( ). If f is a foliated function on M, then we have that

$$\Delta f = \Box''_0 f + Hf,$$

where $\Box''_0 f = -\ g^{\alpha\beta} X_\alpha(X_\beta f) + g^{\alpha\beta}(\nabla_{X_\alpha} X_\beta)_{E^\perp} f$.

Now we consider L^2-transverse field on a complete, non-compact foliated Riemannian manifolds such that the foliation is harmonic and deal with connected and orientable manifolds without boundary.

Let Q^* the dual bundle of Q and its connection be denoted by D^*. Then Q^* has the metric induced from g_Q.

Let $\Lambda^r(M)$ be the space of all r-forms on M and let the exterior derivative $d : \Lambda^r(M) \to \Lambda^{r+1}(M)$ have the formal adjoint operator δ defined by $\delta = (-1)^{(p+q)(r+1)+1} * d* : \Lambda^r(M) \to \Lambda^{r-1}(M)$.

Let $\Gamma_0(Q)$ (resp. $\Gamma_0(Q^*)$) be the space of all sections of Q (resp. Q^*) with compact supports and let $L^2(Q)$ (resp. $L^2(Q^*)$) be the completion of $\Gamma_0(Q)$ with respect to the global scalar product $<< , >>$.

DEFINITION. ([14], [16]) An element $s \in L^2(Q) \cap \Gamma(Q)$ is called an L^2-transverse field of $\mathcal{F}$.

If t is an L^2-transverse field of $\mathcal{F}$, then the dual $\bar{t}$ of t, that is, $\bar{t}(\ ,\ ) = g_Q(t,\ )$ belongs to $L^2(Q^*) \cap \Gamma(Q^*)$.

Let x_0 be a fixed point of M and $\rho(x)$ the geodesic distance from x_0 to $x \in M$. We set

$$B(2k) = \{x \in M | \rho(x) \le 2k\}$$

for any $k > 0$. We consider a differentiable function μ on R which satisfies the following properties :

$$\begin{cases} 0 \le \mu(y) \le 1 & \text{on } \ R, \\ \mu(y) = 1 & \text{for } \ y \le 1, \\ \mu(y) = 0 & \text{for } \ y \ge 2. \end{cases}$$

We define a family $\{\omega_k\}$ of Lischitz continuous functions on M

$$\omega_k(x) = \mu(\frac{\rho(x)}{k}), \quad k = 1, 2, \cdots$$

for any $x \in M$. Then the family $\{\omega_k\}$ has the following properties:

$$\begin{cases} 0 \le \omega_k(x) \le 1 & \text{for } x \in M, \\ supp\omega_k \subset B(2k), & \\ \omega_k(x) = 1 & \text{for } x \in B(k), \\ \lim_{k\to\infty} \omega_k = 1, & \\ |d\omega_k| \le Ck^{-1} & \text{almost everywhere on } M, \end{cases}$$

where C is a positive constant independent of k ([14]).

Let $\widetilde{\Gamma}(Q^*) = \{\eta \in \Gamma(Q^*) | D^*_X\eta = 0 \ for\ any\ X \in \Gamma(E)\}$ and let $\{t_\alpha\}$ be the frame on Q such that $\pi(X_\alpha) = t_\alpha$, and let $\{\tilde{t}_\alpha\}$ be the dual frame to $\{t_\alpha\}$, that is, $\tilde{t}^\alpha(u) = g_Q(t_\alpha, u)$ for all $u \in \Gamma(Q)$. Then we notice that $D^*_X\tilde{t}^\alpha = 0$ for any $X \in \Gamma(E)$. Moreover, we remark that for any $s \in L^2(Q) \cap \overline{V}(\mathcal{F})$ and $\eta \in L^2(Q^*) \cap \widetilde{\Gamma}(Q^*)$, $\omega_k s \to s$ and $\omega_k\eta \to \eta$ as $k \to \infty$ in the strong sense.

The foliation $\mathcal{F}$ is said to be harmonic if $g^{ij}\pi(\nabla^M_{X_i}X_j) = 0$, that is, $\tau = 0$. On the other hand $H = g^{ij}(\nabla^M_{X_i}X_j)_{E^\perp} = g^{ij}$. $\{E^\perp$-component of $\nabla^M_{X_i}X_j\}$ is the mean curvature vector field of each leaf of $\mathcal{F}$. Thus harmonic foliation $\mathcal{F}$ means that all leaves of $\mathcal{F}$ are minimal submanifolds of M.

We finally introduce some lemmas for later use.

LEMMA 4. ([17]) *Let $(M, g_M, \mathcal{F})$ be a complete, non-compact foliated Riemannian manifold with a harmonic foliation $\mathcal{F}$. Then*

$$\int_{B(2k)} div_D(\omega_k s) dS = 0$$

for any $s \in \Gamma(Q)$, where dS denotes the volume element of $B(2k)$.

LEMMA 5. ([1]) *Let $(M, g_M, \mathcal{F})$ be as Lemma 4. Let $s \in \overline{V}(\mathcal{F})$ be an L^2-transverse field of $\mathcal{F}$. Then s is a t.K.f. of $\mathcal{F}$ if and only if $\Delta_D(s) = \rho_D(s)$ and $div_D s = 0$.*

LEMMA 6. ([8]) *Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional connected Riemannian manifold with a harmonic foliation $\mathcal{F}$ of codimension $q \geq 3$ and a complete bundle-like metric g_M with respect to $\mathcal{F}$. Let $s \in \overline{V}(\mathcal{F})$ be an L^2-transverse field of $\mathcal{F}$. Then the following properties are equivalent :*

(1) *s is a transverse Killing field ;*

(2) *s is a transversally divergence-free Jacobi field ;*

(3) *s is a transverse affine field.*

3. Main theorems

Using some lemmas and propositions in the preliminaries, we obtain the following theorems.

THEOREM A. *Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional connected Riemannian manifold with a harmonic foliation $\mathcal{F}$ of codimension $q \geq 2$ and a complete bundle-like metric g_M with respect to $\mathcal{F}$. If an L^2-transverse conformal field s of $\mathcal{F}$ with L^2-charateristic function satisfies $\Theta(s)Ric_D = 0$, then s is a transverse Killing field of $\mathcal{F}$.*

Proof. Let $s \in \overline{V}(\mathcal{F})$ be an L^2-t.c.f. of $\mathcal{F}$ with L^2-characteristic function f_s. Then it follows that

$$g^{\gamma\varepsilon}(\Theta(s)Ric_D)(\pi(X_\gamma), \pi(X_\varepsilon)) = 2(q-1)\Box_0'' f_s$$

(cf. [11, p. 171]). So we obtain directly $\Box_0'' f_s = 0$ by the assumption $\Theta(s)Ric_D = 0$ and $q \geq 2$.

Since $\mathcal{F}$ is harmonic and f_s is a foliated function on M, by means of Proposition 5 we have $\Delta f_s = 0$ so that f_s is constant on M. In fact, we have

$$\begin{aligned}
0 &= << \Delta f_s, \omega_k^2 f_s >>_{B(2k)} \\
&= << \omega_k df_s, \omega_k df_s >>_{B(2k)} + 2 << \omega_k df_s, f_s d\omega_k >>_{B(2k)} \\
&\geq \|\omega_k df_s\|^2_{B(2k)} - 2\|\omega_k df_s\|_{B(2k)} \|f_s d\omega_k\|_{B(2k)} \\
&\geq \frac{3}{4}\|\omega_k df_s\|^2_{B(2k)} - \frac{4C^2}{k^2}\|f_s\|^2_{B(2k)}.
\end{aligned}$$

Since f_s is an L^2-function on M, we have $df_s = 0$ as $k \to \infty$. Therefore f_s is constant on M. Moreover, for any t.c.f. s we have

$$div_D(\omega_k s) = \omega_k q f_s + g_Q(s, grad_D \omega_k)$$

by means of Proposition 1. Thus we obtain

$$\int_{B(2k)} div_D(\omega_k s) dS \geq q f_s \int_{B(2k)} \omega_k dS - \int_{B(2k)} |s||grad_D \omega_k| dS$$
$$\geq q f_s \int_{B(2k)} \omega_k dS - \frac{C}{k} \int_{B(2k)} |s| dS.$$

From Lemma 4, we get $f_s \leq 0$ as $k \to \infty$.

Since $0 = \int_{B(2k)} div_D(\omega_k s) dS \leq q f_s \int_{B(2k)} \omega_k dS + \frac{C}{k} \int_{B(2k)} |s| dS$, we obtain $f_s \geq 0$ as $k \to \infty$. And hence f_s vanishes identically. Thus s is a t.K.f. of $\mathcal{F}$.

THEOREM B. *Let $(M, g_M, \mathcal{F})$ be as Theorem A. If an L^2-transverse projective field s of $\mathcal{F}$ with an L^2-characteristic form ϕ_s satisfies $\Theta(s)$ $Ric_D = 0$, then s is a transverse Killing field of $\mathcal{F}$.*

Proof. For any t.p.f. $s = \pi(Y) \in \overline{V}(\mathcal{F})$ with characteristic form ϕ_s, it follows that

$$g^{\gamma\varepsilon}(\Theta(s) Ric_D)(\pi(X_\gamma), \pi(X_\varepsilon)) = (q-1)\{\delta(\phi_s) - \phi_s(H)\}$$

(cf. [10, p.173]). Since $\mathcal{F}$ is harmonic, $q \geq 2$ and $\Theta(s) Ric_D = 0$, we have $\delta\phi_s = 0$, which together with Proposition 2 implies

$$0 = \delta\phi_s = \frac{1}{q+1} \delta d(div_D s) = \frac{1}{q+1} \Delta(div_D s).$$

Since ϕ_s is an L^2-form on M, $div_D s$ is also an L^2-function on M. If we recall the fact in the process of the proof of Theorem A, then $div_D s$ is constant on M. To use Proposition 2 one more time, we have that $\phi_s = 0$, namely, s is a t.a.f. of $\mathcal{F}$. By means of Proposition 6, we see that s is a t.K.f. of $\mathcal{F}$.

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