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ξ–NULL GEODESIC GRADIENT VECTOR FIELDS ON A LORENTZIAN PARA–SASAKIAN MANIFOLD

KOJI MATSUMOTO, ION MIHAI* AND RADU ROSCA

Introduction

A Lorentzian para-Sasakian manifold $M(\varphi, \xi, \eta, g)$ (abr. LPS-manif old) has been defined and studied in [9] and [10]. On the other hand, para-Sasakian (abr. PS)-manifolds are special semi-cosymplectic manifolds (in the sense of [2]), that is, they are endowed with an almost cosympletic 2-form Ω such that $d^{2\eta}\Omega = \psi(d^w$ denotes the cohomological operator [6]), where the 3-form ψ is the associated Lefebvre form of Ω ([8]). If η is exact, ψ is a $d^{2\eta}$-exact form, the manifold M is called an *exact Ps-manifold*. Clearly, any LPS-manifold is endowed with a semi-cosymplectic structure (abr. SC-structure).

In the present paper, we deal with LPS-manifolds which carries a ξ-null geodesic gradient vector field (abr. ξ-NGG vector field). We recall that the concept of ξ-gradient vector field on a PS-manifold has been recently introduced ([11]) and that on the other hand null geodesics play an important role in different relativistic theories (where there are called light-like geodesics).

Let $M(\varphi, \Omega, \xi, \eta, g)$ be a $(2m+1)$-dimensional LPS-manifold and let ∇, dp and U be a Levi-Civita covariant differential operator with respect to g, the soldering form (or line element) and a real null vector field on M, respectively.

If U satisfies

$$\nabla U = \lambda dp + \eta \otimes U + u \otimes \xi,$$

where λ (resp. $u = b(U)$) is the associated scalar field (resp. the dual form of U), then U is said to be a *ξ-null geodesic gradient vector field* (abr. *$\xi - NGG$ vector field).*

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It is proved that the existence of U is determined by an exterior differential system in involution (in the sense of E. Cartan [3]) and that any manifold M which carries such an U is the local Riemannian product $M = M_U \times M_U^{\perp}$, such that

(i) M_U is a totally geodesic surface of scalar curvature-1, tangent to U and ξ,

(ii) $M_U^{\perp}$ is a totally umbilical 2-codimensional submanifold of M.

The following properties are also proved:

i) U is an exterior concurrent vector field ([11], [13]) and has +1 as conformal scalar;

ii) the conformal scalar λ satisfies

$$\mathrm{Ric}(\varphi U) + \lambda^2 = 0;$$

iii) U defines an infinitesimal contact transformation of η and the necessary and sufficient condition that U be an infinitesimal conformal transformation of Ω, that is,

$$L_U \Omega = r\Omega,$$

is that the conformal scalar r in the above equation be expressed by

$$r = -2\eta(U) + \text{ const.}$$

and in this case U defines an infinitesimal transformation of ψ, too, i.e.,

$$L_U \psi = r\psi,$$

where L_U denotes the Lie derivative with respect to U.

Finally, some properties, when an LSP-manifold M carries in addition of U a null structure conformal vector field C (in the sense of [10]) are also discussed.

1. Preliminaries

Let (M, g) be a Riemannian or pseudo-Riemannian manifold and let ∇ be the covariant differential operator defined by the metric tensor g. We assume in the following that the connection ∇ is symmetric and that M is orientable. Let $\Gamma(TM) = (M)$ (resp. $\flat : TM \to T^*M$) be the set of sections of the tangent bundle TM (resp. the musical isomorphism [12] defined by g). Following W.A.Poor [12], we set

$$A^q(M, TM) = \Gamma \text{ Hom } (\Lambda^q TM, TM)$$

and notice that elements of $A^q(M, TM)$ are vector-valued q-forms ($q < \dim M$). Denote by

$$d^\nabla : A^q(M, TM) \to A^{q+1}(M, TM)$$

the exterior covariant derivative operator with respect to g (generally $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d = 0$) and by $dp \in A^1(M, TM)$ the soldering form of M (see [5]). One has

$$d^\nabla(dp) = 0.$$

A vector field X such that

$$d^\nabla(\nabla X) = \nabla^2 X = \pi \wedge dp \in A^2(M, TM) \tag{1.1}$$

is said to be an *exterior concurrent* (abr. *EC*) *vector field* ([11], [13]) and π is called the *concurrence form* of X. Then one has

$$\pi = f\flat(X), \tag{1.2}$$

where $f \in C^\infty(M)$ is a conformal scalar associated with X. One has

$$S(X, Z) = -(n-1)fg(X, Z); Z \in \Gamma(TM) \tag{1.3}$$

where S means the Ricci tensor field of ∇ and $n = \dim M$.

As a consequence of the above equation, one may write

$$f = \frac{1}{n-1} \text{ Ric}(X), \tag{1.4}$$

where Ric(X) means the Ricci curvature with respect to X.

Any function f such that *gradf* and div(*gradf*) are function of f is called an *isoparametric function* ([14]).

A vector field T such that

$$\nabla T = \lambda dp + \omega \otimes T \tag{1.5}$$

is defined as a *torse forming* ([15]) and if ω is a closed form, then T is called a *closed torse forming.*

The operator

$$d^{\omega} = d + e(\omega) \tag{1.6}$$

acting on ΛM, where $e(\omega)$ means the exterior product by the closed 1-form $\omega \in \Lambda^1 M$, is called the cohomology operator ([6]). Clearly one has

$$d^{\omega} \circ d^{\omega} = 0. \tag{1.7}$$

Any form $\alpha \in \Lambda M$ satisfying $d^{\omega}\alpha = 0$ is said to be d^{ω}*-closed* and if ω is exact, then α is said to be d^{ω}*-exact.*

If C is a conformal vector field (i.e., the conformal version of the Killing equation), one has

$$L_C g = pg \tag{1.8}$$

i.e., $g(\nabla_Z C, Z') + g(\nabla_{Z'} C, Z) = pg(Z, Z')$ and

$$\rho = \frac{2}{n}\ div\ C. \tag{1.9}$$

We recall the following basic formulae associated with C.

$$L_C b(Z) = \rho b(Z) + b([C, Z]), \tag{1.10}$$

$$2L_C S(Z, Z') = \triangle\rho(g(Z, Z')) - (n-2)(Hess_{\nabla}\rho)(Z, Z'), \tag{1.11}$$

$$L_C K = (n-1)\triangle\rho - \rho K, \tag{1.12}$$

where K denotes the scalar curvature of M and the covariant and symmetric 2-tensor $Hess_{\nabla}\rho$ satisfies

$$(Hess_{\nabla}\rho)(Z, Z') = g(Z, H_{\rho}Z') : \ H_{\rho}Z' = \nabla_{Z'}(grad\rho) \tag{1.13}$$

$(see[1])$.

2. ξ-Null geodesic gradient vector fields on an LPS-manifold

Let $M(\varphi, \xi, \eta, g)$ be a $(2m+1)$-dimensional Lorentzian para-Sasakian manifold ([9], [10]). We assume that the metric tensor g is of normal hyperbolic type (see also [4]) and we agree with the following range of indices:

$$A, B = 0, 1, \cdots, 2m, \qquad a, b = 1, 2, \cdots, 2m.$$

Then with respect to an orthonormal vector frame $\{e_A; A = 0, 2m\}$ (abr. 0-basis) one has

$$g(e_A, e_B) = \varepsilon_A \delta_{AB}; \quad \varepsilon_a = -1, \quad \varepsilon_0 = +1. \tag{2.1}$$

Next, by reference to [10], the soldering form dp of M is expressed by

$$dp = -\omega^a \otimes e_a + \eta \otimes \xi; \quad \xi = e_0 \tag{2.2}$$

and E. Cartan's structure equations, with respect to the metric (2.1), are given by

$$\begin{cases} \nabla e_a = \omega_a^{\ b} \otimes e_b + \omega^a \otimes \xi, \\ \nabla \xi = \omega^a \otimes e_a = -dp + \eta \otimes \xi. \end{cases} \tag{2.3}$$

$$\begin{cases} d\omega^a = \omega^b \wedge \omega_b^{\ a} - \eta \wedge \omega^a, \\ d\eta = 0. \end{cases} \tag{2.4}$$

$$d\omega_a^{\ b} = \Omega_a^{\ b} + \omega_a^{\ c} \wedge \omega_c^{\ b} + \omega^{\ a} \wedge \omega^b. \tag{2.5}$$

In the above equations, the 1-form ω^a denote the dual basis with e_a and the 1-form $\omega_a^{\ b}$ (resp. the 2-form $\Omega_a^{\ b}$) are the local connection forms in the bundle $O(M)$ (resp. the curvature forms on M).

On the other hand, the para-Sasakian structure is expressed by the following formulae (see [10])

$$\begin{cases} \varphi^2 = I - \eta \otimes \xi, \quad d\eta = 0, \quad \eta(\xi) = 1, \\ (\nabla \varphi) Z = -b(Z) \otimes \xi - \eta(Z) dp + 2\eta(Z)\eta \otimes \xi; \quad Z \in \Gamma(TM), \\ \nabla_Z \xi = \varphi Z. \end{cases} \tag{2.6}$$

If, in addition, one has

$$\varphi Z = -Z + \eta(Z)\xi, \tag{2.6'}$$

(φ, ξ, η, g) is called a special para-Sasakian (SPS) structure.

We give an example of a Lorentzian para-Sasakian manifolds.

Let R^5 be the 5-dimensional real number space with a coordinate system (x, y, z, t, s). In R^5, if we define

$$\eta = ds - y dx - t dz,$$
$$\xi = \partial/\partial s,$$
$$g = \eta \otimes \eta - dx^2 - dy^2 - dz^2 - dt^2,$$
$$\begin{cases} \phi(\frac{\partial}{\partial_x}) = -\frac{\partial}{\partial_x} - y\frac{\partial}{\partial_s}, \\ \phi(\frac{\partial}{\partial_y}) = -\frac{\partial}{\partial_y}, \\ \phi(\frac{\partial}{\partial_z}) = -\frac{\partial}{\partial_z} - t\frac{\partial}{\partial_s} \\ \phi(\frac{\partial}{\partial_t}) = -\frac{\partial}{\partial_t},\ \phi(\frac{\partial}{\partial_s}) = 0, \end{cases}$$

then (ϕ, ξ, η, g) is a Lorentzian para-Sasakian structure in R^5. Then the metric tensor g is expressed by

$$g = \begin{pmatrix} -1+y^2 & 0 & ty & 0 & -y \\ 0 & -1 & 0 & 0 & 0 \\ ty & 0 & -1+t^2 & 0 & -t \\ 0 & 0 & 0 & -1 & 0 \\ -y & 0 & -t & 0 & 1 \end{pmatrix}$$

In [10], it has been proved that operating by d^∇ on $\nabla\xi$ one has

$$\nabla^2\xi = \eta \wedge dp \tag{2.7}$$

and the above equation shows that as in the Riemannian case, ξ is an EC-vector field with $+1$ as conformal scalar.

It should also be noticed that by (2.6′) one deduces the following relation

$$\varphi dp + dp = \eta \otimes \xi \Rightarrow b(\varphi Z) + b(Z) = \eta(Z)\eta. \tag{2.8}$$

On the other hand, since ξ is an EC-vector field, then by reference to [11], the Ricci tensor $S(\xi, Z)$ is given by

$$S(\xi, Z) = -2mg(\xi, Z); \quad Z \in \Gamma(TM) \tag{2.9}$$

Let now U be a null real vector field on M. We agree with the following definition:

The vector field U is said to be a *ξ-null gradient geodesic* (abr. $\xi - NGG$) vector field if its covariant differential ∇U satisfies

$$\nabla U = \lambda dp + \eta \otimes U + u \otimes \xi, \tag{2.10}$$

where $\lambda \in C^\infty M$ and $u = b(U)$.

Since, u is the dual form of U, we have $u(U) = g(U, U) = 0$ and from (2.10) we get

$$\nabla_U U = (\lambda + \eta(U))U \tag{2.11}$$

and

$$g(\nabla_Z U, Z') = g(\nabla_{Z'} U, Z); \quad Z, Z' \in \Gamma(TM) \tag{2.12}$$

which show that U is a null geodesic and a gradient vector field, respectively. From (2.10) one derives

$$\lambda + \eta(U) = 0 \Rightarrow \nabla_U U = 0 \tag{2.13}$$

and

$$d\eta(U) = \eta(U)\eta \Rightarrow \frac{d\lambda}{\lambda} = \eta. \tag{2.14}$$

Following a known definition, the equation (2.13) expresses that U is a strict geodesic.

Setting $U = U^0\xi + U^a e_a$, we find $\eta(U) = U^0$, and since by (2.14) it is seen that η is an exact form, we conclude that if LSP-manifold carries a ξ-NGG vector field, then it is an exact LSP-manifold.

Taking account of (2.6), one gets at once by (2.13) and (2.14)

$$L_U \eta = -\lambda\eta. \tag{2.15}$$

Hence, using a known definition, U defines an infinitesimal contact transformation of the paracontact sturcture of M.

Since, $u = b(U)$ is expressed by

$$u = U^0\eta + \sum_a U^a\omega^a,$$

then by (2.10) and (2.14) and with the help of (2.4) one checks that u is a closed form;

$$du = 0. \tag{2.16}$$

If we put $v = b(\varphi U)$, it follows from (2.8), (2.14) and (2.16) that

$$dv = 0 \tag{2.17}$$

and hence the vector field φU is also a closed vector field.

Operating now on ∇U by the exterior covariant derivative opertor d^∇ and taking account of (2.3), (2.14) and (2.16) one derives

$$d^\nabla(\nabla U) = \nabla^2 U = u \wedge dp. \tag{2.18}$$

Therefore by reference to (1.1) the above equation proves the striking fact that U is an EC-vector field.

We recall that the property of exterior concurrency is invariant under the action of φ. One may easily check, using (2.6′), that φU is also an EC-vector field, i.e.

$$\nabla^2\varphi U = v \wedge dp. \tag{2.19}$$

Using (2.18) and (2.19), (1.3) becomes

$$S(U, Z) = -2mg(U, Z) \tag{2.20}$$

and similarly we have

$$S(\varphi U, Z) = -2mg(\varphi U, Z). \tag{2.21}$$

It follows that $\mathrm{Ric}(\varphi U) = -(U^0)^2 = -\lambda^2$.

Since $\eta(\varphi U) = 0$, then clearly $S(\varphi U, \xi) = 0$.

By (2.6) and (2.10) one finds

$$\nabla \varphi U = -\eta \otimes U + (v + U^0 \eta) \otimes \xi \tag{2.22}$$

and on behalf of (2.3) one gets

$$[\xi, \varphi U] = 2\varphi U$$

which shows that φU admits infinitesimal transformations of generators ξ (see also [4]).

Denote now by $\sum$ the exterior differential system which defines the vector field U. By (2.14), (2.16) and (2.17) it is seen that the chracteristic numbers of $\sum$ are $r = 3$, $s_0 = 1$, $s_1 = 2$. Therefore, by E.Cartan's test ([3]), $\sum$ is in involution (i.e. $r = s_0 + s_1$) and the existence of $\sum$ is determined by two arbitrary functions of one argument.

Next let $D_U = \{U, \xi\}$ be the 2-distribution spanned by U and ξ. By (2.3) and (2.10) it is easily seen that $\nabla_{U''} U' \in D_U$, where U', U'' are any vector fields of D_U. In consequence of this fact and by virtue of a known result (see also [7]), D_U is an autoparallel foliation. If we denote by M_U the leaves (surfaces) of D_U, then as is known ([7] M_U are totally geodesic submanifolds of M.

On the other hand, since the property of exterior concurrency is invariant by linearisation it follows that any vector field on M_U is EC. In consequence of this fact and the general properties of EC-vector fields ([11]), we conclude that M_U is of scalar curvature-1.

By (2.13) and (2.14) one may write

$$grad\lambda = \lambda\xi \tag{2.23}$$

and since it is known ([10]) that $\operatorname{div}\xi = -2m$, we deduce

$$\operatorname{div}(grad\lambda) = -(2m-1)\lambda \tag{2.24}$$

and from the definition of the Laplace operator $\triangle\nu = -\operatorname{div}(grad\nu)$, one has $\triangle\lambda = (2m-1)\lambda$, which proves that the conformal scalar λ associated with U is an eigenfunction of $\triangle$.

On the other hand, since ξ is a time like vector field, it follows instantly by (2.23)

$$\|grad\lambda\|^2 = \lambda^2 \tag{2.25}$$

which on behalf of (2.24) proves that λ is an isoparametric function ([14]).

By setting $T = \lambda\xi = grad\lambda$, after a short calculation we obtain from (2.3) that

$$\nabla T = -dp + \eta \otimes T$$

which shows that T defines a closed torse forming (see (1.5)).

THEOREM. *Let $M(\varphi, \xi, \eta, g)$ be a Lorentzian special para-Sasakian manifold of dimension $2m+1$ and let U be a ξ-null geodesic gradient vector field on M. The existence of U is determined by an exterior differential system in involution and any M which carries such a null vector field U is the local Riemannian product*

$$M = M_U \times M_U^{\perp}$$

such that

i) *M_U is a totally geodesic surface of scalar curvature -1 tangent to U and ξ:*

ii) *$M_U^{\perp}$ is a totally umbilical 2-codimensional submanifold having U as normal null section.*

Furthermore:

i) *U is an exterior concurrent vector field:*

ii) *the conformal scalar λ associated with U is an isoparametric function and satisfies*

$$Ric(\varphi U) + \lambda^2 = 0;$$

iii) *U defines an infinitesimal contact transformation on M and φU admits infinitesimal transformations of generators ξ.*

3. Infinitesimal conformal transformations on an LPS-manifold

We recall that in [2] it has been proved that any PS-manifold is endowed with a semi-cosymplectic structure defined by the pairing (Ω, η), where Ω is a 2-form of rank $2m$ and such that

$$\Omega^m \wedge \eta \neq 0, \quad d^{2\eta}\Omega = \psi, \quad d\eta = 0. \tag{3.1}$$

In (3.1), d^ω denotes the cohomological operator (see (1.6)) and ψ is a 3-form associated with Ω and called the Lefebvre form. Clearly any LPS-manifold is endowed with semi-cosymplectic structure. If, as in [2], we consider the globally defined 2-form

$$\Omega = \omega^i \wedge \omega^{i^*} : \quad i = 1, m; \; i^* = i + m, \tag{3.2}$$

it is easily seen with the help of (2.4) that one has

$$d^{2\eta}\Omega = d\Omega + 2\eta \wedge \Omega = \psi \Rightarrow d^{2\eta}\psi = 0 \tag{3.3}$$

and since η is exact one may say that the Lefebvre form ψ is $d^{2\eta}$-exact.

Assume now that U defines an infinitesimal conformal transformation (abr. ICT) of Ω, that is,

$$L_U\Omega = r\Omega \tag{3.4}$$

where r is a conformal scalar. Since L and d commute, one derives from (3.3) and (3.4)

$$L_U(-2\eta \wedge \Omega + \psi) = dr \wedge \Omega + r(\psi - 2\eta \wedge \Omega) \tag{3.5}$$

or equivalently

$$L_U\psi - 2U^0\eta \wedge \Omega - 2r\eta \wedge \Omega = dr \wedge \Omega + r\psi - 2r\eta \wedge \Omega.$$

Hence the necessary and sufficient condition that (3.4) holds good is

$$dr = -2U^0\eta = 2\lambda\eta$$

that is by (2.14)

$$r + 2U^0 = const, \quad \text{or} \quad r = 2\lambda + const. \tag{3.6}$$

and in this case U defines also an ICT of ψ, i.e.

$$L_U\psi = r\psi \tag{3.7}$$

Assume now that the LPS-manifold under consideration carries in addition of the null vector field U a null structure conformal vector field C (in the sence of [10]). By reference to [10], C is define by

$$\nabla C = f dp + \xi \wedge C, \quad g(C, C) = 0. \tag{3.8}$$

If $\alpha = b(C)$ denotes the dual form of C, one obtains from (3.8)

$$d^{2\eta}\alpha = 0, \tag{3.9}$$

$$dC^0 = f\eta; \quad C^0 + f = 0 \tag{3.10}$$

and as is known the conformal scalar ρ associated with C (i.e. $L_C g = \rho g$) is given by $\rho = 2f$.

By (3.9), (3.10) and (2.14) we see that α is as ψ a $d^2\eta$-exact form and that the conformal scalars λ and ρ associated with the null vector fielde U and C respectively are related by $\rho = c/\lambda$; $c = const.$.

Next by (2.13) one gets

$$[C, U] = -fU \tag{3.11}$$

and making use of (1.10), one finds

$$L_C u = 2fu + b([C, U]) = fu = \frac{\rho}{2}u \tag{3.12}$$

which shows that C defines a ICT of u.

By setting now $s = g(C, U)$, one deduces from (2.10) and (3.8)

$$ds = -s\eta \tag{3.13}$$

and so taking account of (2.13), one gets by (3.9)

$$(3.14) \qquad L_U\alpha = 2\lambda\alpha + s\eta.$$

By exterior differentiation we obtain $d^\eta(L_U\alpha) = 0$.

Therefore one may say that the Lie derivative $L_U\alpha$ is d^η-exact. Finally denote by $D_C = \{C, U, \xi\}$ the 3-dimensional distribution spanned by the vector fields C, U and ξ.

Since C is an EC-vector field and

$$\nabla C = fdp + \xi \wedge C = fdp + \alpha \otimes \xi - \eta \otimes C,$$

from (2.3) and (2.10) one has $\nabla_{C''}C' \in D_C$, for any $C', C'' \in D_C$. Thus we may state the following property: Any LPS-manifold M which carries a ξ-NGG vector field U and a null structure conformal vector field C is the local Riemannian product

$$M = M_C \times M_C^\perp$$

such that:

i) M_C is a 3-dimensional totally geodesic submanifold of scalar curvature -1, tangent to C, U and ξ;

ii) $M_C^\perp$ is a totally umbilical 3-codimensional submanifold, having C and U as null normal sections.

Summing up, we may formulate the following

THEOREM. *Let $M(\varphi, \Omega, \xi, \eta, g)$ be a $(2m+1)$-dimensional LSPS-manifold having Ω as almost cosymplectic form and let ψ be the Lefebvre form associated with the semi-cosymplectic structure defined by the pairing (Ω, η). Suppose that M carries a ξ-NGG vector field U.*

Then the necessary and sufficient condition in order that U define an infinitesimal conformal transformation of Ω, i.e. $L_U\Omega = r\Omega$, is that the conformal scalar r be defined by $r = -2\eta(U) + const.$ and in this case U defines also an ICT of ψ, i.e. $L_U\psi = r\psi$.

If in addition of U, M carries a null structure conformal vector field C, then M is the local Riemannian product $M = M_C \times M_C^\perp$ such that

i) M_C is a 3-dimensional submanifold of scalar curvature -1 and it is totally geodesic and tangent to C, U and ξ;

ii) $M_C^{\perp}$ is totally umbilical 3-codimensional submanifold.

Furthermore, the conformal scalar ρ and λ corresponding to C and U respectively, satisfy $\rho\lambda = const.$ and a) C defines an infinitesimal conformal transformation of the dual form of U;

b) the Lie derivative with respect to U of the dual form of C is d^{η}-exact.

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Koji Matsumoto
Department of Mathematics
Faculty of Education

Yamagata University
Yamagata 990, Japan

Ion Mihai
Faculty of Mathematics
University of Bucharest
Str. Academiei Nr.14
Bucharest, Romania

Radu Rosca
59 Avenue Emile Zola
75015 Paris, France