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EPIMORPHISMS OF ANNIHILATORS OF POOR M–COSEQUENCES

SANG-CHO CHUNG

0. Introduction

Let R be a commutative ring with identity and M an R-module.

In ([Mt], 8), Matlis proved that, for a given M-sequence $\{x_1, \ldots, x_n\}$, the following map

$$M/({x_1}^t, \ldots, {x_n}^t)M \longrightarrow M/({x_1}^{t+1}, \ldots, {x_n}^{t+1})M$$

is a monomorphism for all $t > 0$, and if $\{x_1, \ldots, x_n\}$ is an M-cosequence, then

$$\mathrm{Ann}_M({x_1}^{t+1}, \ldots, {x_n}^{t+1})R \longrightarrow \mathrm{Ann}_M({x_1}^t, \ldots, {x_n}^t)R$$

is an epimorphism for all $t > 0$.

As a generalization of the first result of Matlis, in ([O], 3.2), O'carroll described that, when $\{y_1, \ldots, y_n\}$ is a poor M-sequence and $\{x_1, \ldots, x_n\}$ is a sequence of elements of R such that $\mathrm{H}[x_1 \ \ldots \ x_n]^T = [y_1 \ \ldots \ y_n]^T$ for some $n \times n$ lower triangular matrix H, the map

$$M/(x_1, \ldots, x_n)M \longrightarrow M/(y_1, \ldots, y_n)M$$

is a monomorhism and $\{x_1, \ldots, x_n\}$ is also a poor M-sequence.

So we consider the dual case of O'carroll. That is, let $\{y_1, \ldots, y_n\}$ be a poor M-cosequence and $\{x_1, \ldots, x_n\}$ is a sequence of elements of R such that $\mathrm{H}[x_1 \ \ldots \ x_n]^T = [y_1 \ \ldots \ y_n]^T$ for some $n \times n$ lower triangular matrix H. We give an epimorphism

$$\mathrm{Ann}_M(y_1, \ldots, y_n)R \longrightarrow \mathrm{Ann}_M(x_1, \ldots, x_n)R$$

and $\{x_1, \ldots, x_n\}$ is also a poor M-cosequence.

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1. Preliminaries

Throughout this note, R is a commutative ring with identity and M an R-module. We use ^{T} to denote matrix transpose and $D_n(R)$ ($n \geq 1$) to denote the set of $n \times n$ lower triangular matrices over R. For $\mathrm{H} \in D_n(R)$, $|\mathrm{H}|$ denotes the determinant of H. Let $(a_1, \ldots, a_i)R$ be the ideal of R generated by $\{a_1, \ldots, a_i\}$ and $(a_1, \ldots, a_i)M$ the submodule of M generated by $\{a_j m : j = 1, \ldots, i \text{ and } m \in M\}$.

Let $\{x_1, \ldots, x_n\}$ be a sequence of elements of R and M an R-module. Then $\{x_1, \ldots, x_n\}$ is said to be a *poor M-sequence* if multiplication by x_i on $M/(x_1, \ldots, x_{i-1})M$ is a monomorphism for all $i = 1, \ldots, n$ (where $x_0 = 0$). If, in addition, $M/(x_1, \ldots, x_n)M \neq 0$, we call $\{x_1, \ldots, x_n\}$ an *M-sequence.*

If $\mathfrak{b}$ is an ideal of R, we define $\mathrm{Ann}_M \mathfrak{b} = \{m \in M : \mathfrak{b}m = 0\}$. We have a dual definition; $\{x_1, \ldots, x_n\}$ is said to be a *poor M-cosequence* if multiplication by x_i on $\mathrm{Ann}_M(x_1, \ldots, x_{i-1})R$ is an epimorphism for all $i = 1, \ldots, n$ (where $x_0 = 0$). Similarly, if $\mathrm{Ann}_M(x_1, \ldots, x_n)R \neq 0$, $\{x_1, \ldots, x_n\}$ is called an *M-cosequence.*

Let E be an injective envelope of the direct sum of all of the simple R-modules, and define the functor * by $^* = \mathrm{Hom}(-, E)$, then * is a faithfully exact contravariant functor; that is, a sequence of R-modules is exact if and only if its * is exact.

LEMMA 1.1. *Let R be a ring and M an R-module. Assume that N is an injective R-module and $\mathfrak{a}$ a finitely generated ideal of R. Then we have the following.*

(1) $R/\mathfrak{a} \otimes_R \mathrm{Hom}(M, N) \cong \mathrm{Hom}(\mathrm{Hom}(R/\mathfrak{a}, M), N)$.

(2) *If, in addition, $\mathfrak{a}$ is generated by a poor M-cosequence and $N = E$, then we have*

$$\mathfrak{a} \otimes_R \mathrm{Hom}(M, E) \cong \mathrm{Hom}(\mathrm{Hom}(\mathfrak{a}, M), E) \cong \mathfrak{a}M^*.$$

(3) *If, in addition, $\mathfrak{a}$ is generated by a poor M-sequence, then we have*

$$\mathfrak{a} \otimes M \cong \mathfrak{a}M.$$

In particular, we have $\mathrm{Hom}(\mathfrak{a} \otimes M, E) \cong \mathrm{Hom}(\mathfrak{a}, \mathrm{Hom}(M, E)) \cong (\mathfrak{a}M)^*$.

Proof. (1) Consider the following exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow R \longrightarrow R/\mathfrak{a} \longrightarrow 0.$$

Then by ([Mm], p.14 Exercise 2.5(b)) $R/\mathfrak{a}$ is of finite presentation. Hence the assertion follows from ([R], 3.60).

(2) Assume that $\mathfrak{a}$ is generated by an M-cosequence. Note that the generators of $\mathfrak{a}$ forms an M^*-sequence by ([Mt], 5(2)) or the following Lemma 1.2(2). From the above short exact sequence, we have the following short exact sequence;

$$0 \longrightarrow \mathrm{Hom}(R/\mathfrak{a}, M) \longrightarrow \mathrm{Hom}(R, M) \longrightarrow \mathrm{Hom}(\mathfrak{a}, M) \longrightarrow 0,$$

since $\mathrm{Ext}^1_R(R/\mathfrak{a}, M) = 0$ by ([M_t], 4 and [BH], 1.1.12).

Hence we get the following exact sequence;

$$0 \longrightarrow \mathrm{Hom}(\mathrm{Hom}(\mathfrak{a}, M), E) \longrightarrow \mathrm{Hom}(M, E) \\ \longrightarrow \mathrm{Hom}(\mathrm{Hom}(R/\mathfrak{a}, M), E) \longrightarrow 0,$$

since E is an injective R-module.

Now, from (1) and the short exact sequence in the proof of (1), we have the following commutative diagram;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{a} \otimes \mathrm{Hom}(M, E) & \longrightarrow & R \otimes \mathrm{Hom}(M, E) & \longrightarrow & R/\mathfrak{a} \otimes \mathrm{Hom}(M, E) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \| & & \\ 0 & \longrightarrow & \mathrm{Hom}(\mathrm{Hom}(\mathfrak{a}, M), E) & \longrightarrow & \mathrm{Hom}(M, E) & \longrightarrow & \mathrm{Hom}(\mathrm{Hom}(R/\mathfrak{a}, M), E) & \longrightarrow & 0, \end{array}$$

since $\mathrm{Tor}_1^R(R/\mathfrak{a}, \mathrm{Hom}(M, E)) = 0$ by ([BH], 1.1.12). Hence the five lemma gives the first isomorphism.

Next since $R \otimes \mathrm{Hom}(M, E) \cong M^*$ and $R/\mathfrak{a} \otimes \mathrm{Hom}(M, E) \cong M^*/\mathfrak{a}M^*$, we have $\mathfrak{a} \otimes \mathrm{Hom}(M, E) \cong \mathfrak{a}M^*$ from the top exact sequence of the above commutative diagram.

(3) Using the short exact sequence in the proof of (1) again, we have the following short exact sequence;

$$0 \longrightarrow \mathfrak{a} \otimes M \longrightarrow M \longrightarrow M/\mathfrak{a}M \longrightarrow 0,$$

since $\mathrm{Tor}_1^R(R/\mathfrak{a}, M) = 0$ by ([BH], 1.1.12). Hence we obtain $\mathfrak{a} \otimes M \cong \mathfrak{a}M$.

LEMMA 1.2. (cf. [Mt], 1, 5 and 6) (1) $\{x_1, \ldots, x_n\}$ *is a poor M-sequence if and only if* $\{x_1, \ldots, x_n\}$ *is a poor* M^**-cosequence.*
(2) $\{x_1, \ldots, x_n\}$ *is a poor M-cosequence if and only if* $\{x_1, \ldots, x_n\}$ *is a poor* M^**-sequence.*
(3) $\{x_1, \ldots, x_n\}$ *is a poor M-sequence if and only if* $\{x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}\}$ *is a poor M-sequence for any positive integers* $\alpha_1, \ldots, \alpha_n$.
(4) $\{x_1, \ldots, x_n\}$ *is a poor M-cosequence if and only if* $\{x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}\}$ *is a poor M-cosequence for any positive integers* $\alpha_1, \ldots, \alpha_n$.

Proof. ((1) and (2)) Note that for $i = 1, \ldots, n$

$$\begin{aligned}(M/(x_1, \ldots, x_i)M)^* &\cong \mathrm{Hom}(R/(x_1, \ldots, x_i)R \otimes M, E) \\ \cong &\mathrm{Hom}(R/(x_1, \ldots, x_i)R, \mathrm{Hom}(M, E)) \cong \mathrm{Ann}_{M^*}(x_1, \ldots, x_i)R\end{aligned}$$

and

$$\begin{aligned}(\mathrm{Ann}_M(x_1, \ldots, x_i)R)^* &\cong \mathrm{Hom}(\mathrm{Hom}(R/(x_1, \ldots, x_i)R, M), E) \\ \cong &R/(x_1, \ldots, x_i)R \otimes \mathrm{Hom}(M, E) \cong M^*/(x_1, \ldots, x_i)M^*\end{aligned}$$

by Lemma 1.1(1).

Then the results follow easily from the above isomorphisms, since * is faithfully exact.

(3) This follows from ([K], p.102 Exercise 12).

(4) The proof follows immediately from (2) and (3).

2. Main results

LEMMA 2.1. ([O], 3.2) *Let R be a ring and M an R-module. Consider two sequences* $\{x_1, \ldots, x_n\}$ *and* $\{y_1, \ldots, y_n\}$ *of elements of R such that*

(i) $\mathrm{H}[x_1 \ \ldots \ x_n]^T = [y_1 \ \ldots \ y_n]^T$ *for some* $\mathrm{H} \in D_n(R)$, *and*

(ii) $\{y_1, \ldots, y_n\}$ *is a poor M-sequence.*

Then the map from $M/(x_1, \ldots, x_n)M$ *to* $M/(y_1, \ldots, y_n)M$ *induced by multiplication by* $|\mathrm{H}|$ *is a monomorphism and* $\{x_1, \ldots, x_n\}$ *is also a poor M-sequence.*

THEOREM 2.2. *Let R be a ring and $M \in R$-module. Consider two sequences $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ of elements of R such that*
(i) $\mathrm{H}[x_1 \ \ldots \ x_n]^T = [y_1 \ \ldots \ y_n]^T$ for some $\mathrm{H} \in D_n(R)$, and
(ii) $\{y_1, \ldots, y_n\}$ is a poor M-cosequence.
Then the map from $\mathrm{Ann}_M(y_1, \ldots, y_n)R$ to $\mathrm{Ann}_M(x_1, \ldots, x_n)R$ induced by multiplication by $|\mathrm{H}|$ is an epimorphism and $\{x_1, \ldots, x_n\}$ is also a poor M-cosequence.

Proof. We first prove that the map is well defined by induction on n. Suppose that $n = 1$, and that $\mathrm{H} = (h)$ with $y_1 = hx_1$. Then for all $m \in \mathrm{Ann}_M(y_1)$, *i.e.*, $my_1 = 0$, and $mhx_1 = 0$. Hence $|\mathrm{H}|m \in \mathrm{Ann}_M(x_1)$.

Assume that it is true when $n-1$. Let $m \in \mathrm{Ann}_M(y_1, \ldots, y_n)R$. Then we have $m \in \mathrm{Ann}_M(y_1, \ldots, y_{n-1})R$ and $\mathrm{H}'[x_1 \ \ldots \ x_{n-1}]^T = [y_1 \ \ldots \ y_{n-1}]^T$ where H' is the top left $(n-1) \times (n-1)$ submatrix of H. Hence by inductive hypothesis we have

$$h_{11} \cdots h_{n-1,n-1} m \in \mathrm{Ann}_M(x_1, \ldots, x_{n-1})R.$$

Since $m \in \mathrm{Ann}_M(y_n)$, we get $my_n = m(\sum_{j=1}^n h_{nj}x_j) = 0$. Therefore we have

$$h_{11} \cdots h_{n-1,n-1} m (\sum_{j=1}^{n-1} h_{nj}x_j + h_{nn}x_n) = 0$$

or

$$h_{11} \cdots h_{nn} m x_n = 0.$$

Hence
$|\mathrm{H}|m \in \mathrm{Ann}_M(x_1, \ldots, x_{n-1})R \cap \mathrm{Ann}_M(x_n) = \mathrm{Ann}_M(x_1, \ldots, x_n)\, R.$

Now, we consider the following exact sequence;

$$\mathrm{Ann}_M(y_1, \ldots, y_n)R \xrightarrow{|\mathrm{H}|} \mathrm{Ann}_M(x_1, \ldots, x_n)R \longrightarrow C \longrightarrow 0,$$

so that

$$\mathrm{Hom}(R/(y_1, \ldots, y_n)R, M) \xrightarrow{|\mathrm{H}|} \mathrm{Hom}(R/(x_1, \ldots, x_n)R, M) \longrightarrow C \longrightarrow 0.$$

Hence we have the following exact sequence;

$$0 \longrightarrow \mathrm{Hom}(C,E) \longrightarrow \mathrm{Hom}(\mathrm{Hom}(R/(x_1,\dots,x_n)R,M),E) \longrightarrow \mathrm{Hom}(\mathrm{Hom}(R/(y_1,\dots,y_n)R,M),E).$$

By Lemma 1.1(1), we obtain

$$0 \longrightarrow \mathrm{Hom}(C,E) \longrightarrow R/(x_1,\dots,x_n)R \otimes_R M^* \longrightarrow R/(y_1,\dots,y_n)R \otimes_R M^*.$$

That is,

$$0 \longrightarrow \mathrm{Hom}(C,E) \longrightarrow M^*/(x_1,\dots,x_n)M^* \longrightarrow M^*/(y_1,\dots,y_n)M^*.$$

Since $\{y_1,\dots,y_n\}$ is a poor M^*-sequence by Lemma 1.2(2), we have $\mathrm{Hom}(C,E)=0$ and $\{x_1,\dots,x_n\}$ is a poor M^*-sequence by Lemma 2.1. Hence we get $C=0$, since $\mathrm{Hom}(-,E)$ is faithfully exact.

COROLLARY 2.3. (cf. $[M_t]$, 8) (1) *If $\{x_1,\dots,x_n\}$ is a poor M-seque nce, then*

$$\alpha_t : M/(x_1{}^t,\dots,x_n{}^t)M \xrightarrow{\underline{x}} M/(x_1{}^{t+1},\dots,x_n{}^{t+1})M$$

defined by $\alpha_t(m+(x_1{}^t,\dots,x_n{}^t)M)=\underline{x}\cdot m+(x_1{}^{t+1},\dots,x_n{}^{t+1})M$ with $\underline{x}=x_1\cdots x_n$ is a monomorphism for all $t>0$.

(2) *If $\{x_1,\dots,x_n\}$ is a poor M-cosequence, then*

$$\beta_t : \mathrm{Ann}_M(x_1{}^{t+1},\dots,x_n{}^{t+1})R \xrightarrow{\underline{x}} \mathrm{Ann}_M(x_1{}^t,\dots,x_n{}^t)R$$

induced by multiplication by $\underline{x}=x_1\cdots x_n$ is an epimorphism for all $t>0$.

Proof. From Lemma 1.2(3)(4), we have $\{x_1{}^{\alpha_1},\dots,x_n{}^{\alpha_n}\}$ is a poor M-sequence (-cosequence) for any positive integers $\alpha_1,\dots,\alpha_n$.

Hence consider that H is a diagonal matrix $diag(x_1,\dots,x_n)$ such that

$$\mathrm{H}[x_1{}^t \ \dots \ x_n{}^t]^T = [x_1{}^{t+1} \ \dots \ x_n{}^{t+1}]^T.$$

Then the corollary follows easily from Lemma 2.1 (Theorem 2.2).

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Department of Mathematics, Chungnam National University, Taejon 305-764, Korea