Bull. Korean Math. Soc. 32 (1995), No. 2, pp. 349–357

DRINFELD MODULES WITH BAD REDUCTION OVER COMPLETE LOCAL RINGS

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0. Introduction

In the theory of elliptic curves over a complete field with bad reduction (i.e. with nonintegral j-invariant) Tate elliptic curves play an important role. Likewise, in the theory of Drinfeld modules, Tate-Drinfeld modules replace Tate elliptic curves.

In this note we define the Hasse invariant of a rank 2 Drinfeld module on $\mathbb{F}_q[T]$ defined over a field K. As in the classical theory of elliptic curves, the j-invariant and the Hasse invariant together determine a K-isomorphism class of Drinfeld modules of rank 2. Using the Fourier expansions of g, Δ, and j we obtain a criterion for a Drinfeld module to be K-isomorphic to a Tate-Drinfeld module (Theorem 1.2). Then we prove an Isogeny Theorem for Drinfeld modules with non-integral j-invariants (Theorem 2.2). Finally we investigate the torsion points of a Tate-Drinfeld module. Using this information we obtain an analogue of the Theorem of Kodaira-Neron (Theorem 3.1) and a proof of an analogue of the Neron-Ogg-Shafarevich criterion (Theorem 3.2).

1. Tate-Drinfeld Modules

Let $A = \mathbb{F}_q[T]$ and ρ be the Carlitz module. Throughout the paper K denote a complete field with respect to a discrete valuation v, R its ring of integers, and k the residue field. By a Drinfeld module we always mean a Drinfeld module of rank two. Let $g(t)$ and $\Delta(t)$ be the

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Received June 23, 1994.

1991 AMS Subject Classification: 11G09.

Key words: Drinfeld Module, Tate-Drinfeld Module, Hasse Invariant.

Partially supported by Non Directed Research Fund, Korea Research Foundation, 1993.

standard modular forms of weight $q-1$ and q^2-1, respectively. Then they are the power series in $s = t^{q-1}$. Let $\tilde{g}(s)$ and $\tilde{\Delta}(s)$ be the power series given by

$$\tilde{g}(t^{q-1}) = g(t), \qquad \tilde{\Delta}(t^{q-1}) = \Delta(t).$$

Now for an element s of K with $|s| < 1$, le ϕ^s be the Drinfeld module given by

$$\phi^s_T = T\tau^0 + \tilde{g}(s)\tau + \tilde{\Delta}(s)\tau^2.$$

We call ϕ^s the *Tate-Drinfeld module* associated to s and call s the *period* of the Tate-Drinfeld module ϕ^s. It is well-known that the *j-invariant*, j, of ϕ^s is of the form

$$(*) \qquad j = \frac{1}{s} + f(s)$$

where f is a power series with coefficients in A. Equivalently,

$$s = \frac{1}{j} + h\left(\frac{1}{j}\right)$$

where h is a power series with coefficients in A and of height at least 2. In particular, $|j| > 1$.

Let t be an element of K^{ac} with $t^{q-1} = s$. Define $e_t(u)$ by,

$$e_t(u) = u \cdot \prod_{a \in A - \{0\}} \left(1 - \frac{u}{\rho_a(t^{-1})}\right).$$

It is easy to see that $e_t(u)$ is defined over K, and so we write it by $e_s(u)$. Then we have the following commutative diagram with exact rows (cf: [1] for details);

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D_t & \longrightarrow & \tilde{K} & \xrightarrow{e_s} & \tilde{K} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho_a} & & \downarrow{\scriptstyle \rho_a} & & \downarrow{\scriptstyle \phi^s_a} & & \\
0 & \longrightarrow & D_t & \longrightarrow & \tilde{K} & \xrightarrow{e_s} & \tilde{K} & \longrightarrow & 0
\end{array}$$

where $\tilde{K} = K(t)$ and $D_t = \{\rho_a(t^{-1}) : a \in A\}$. Here D_t plays the role of an A-lattice in K with the A-module structure via ρ.

We say that $s = t^{q-1}$ and $s' = t'^{q-1}$ in K^* are *commensurable* if there exist a and b in A such that $\rho_a(t^{-1}) = \rho_b(t'^{-1})$. As with the usual theory of Drinfeld modules, we have a natural bijection (cf; [4] Proposition 3.8)

$$Hom(\phi^s, \phi^{s'}) \simeq \{\rho_a : \rho_a(D_t) \subset D_{t'}\}.$$

Thus ϕ^s and $\phi^{s'}$ are isogenous if and only if s and s' are commensurable. The morphism $\mathfrak{f}_a$ of $Hom(\phi^s, \phi^{s'})$ associated to ρ_a is given by

$$\mathfrak{f}_a(u) = au \cdot \prod_{\alpha \in \rho_a^{-1}(D_{t'})/D_t - \{0\}} \left(1 - \frac{u}{e_s(\alpha)}\right),$$

where $\rho_a^{-1}(D_{t'}) = \{\alpha \in K^a : \rho_a(\alpha) \in D_{t'}\}$. Since ρ_a, e_s, and $e_{s'}$ are defined over K, $\mathfrak{f}_a$ is also defined over K. Since $\mathfrak{f}_a \circ e_s$ and $e_{s'} \circ \mathfrak{f}_a$ both have simple zeros at the pointsof $\rho_a^{-1}(D_{t'})$, we have

$$\mathfrak{f}_a \circ e_s = e_{s'} \circ \mathfrak{f}_a$$

by comparing the coefficients of the starting terms.

PROPOSITION 1.1. *Assume that* $\rho_a(t^{-1}) = \rho_b(t'^{-1})$. *Let* $\mathfrak{f}_a$ *(resp.* $\mathfrak{f}'_b$*) be the element of* $Hom(\phi^{t'}, \phi^t)$ *(resp.* $Hom(\phi^t, \phi^{t'})$*) associated to* ρ_a *(resp.* ρ_b*). Then we have*

(i) $\mathfrak{f}'_b \circ \mathfrak{f}_a = \phi^t_{ab}$

(ii) $\mathfrak{f}_a \circ \mathfrak{f}'_b = \phi^{t'}_{ab}$

(iii) $\deg \mathfrak{f}_a = deg\, f'_b = q^{deg\, a + deg\, b}$.

Proof. (i) and (ii) are trivial from the construction. Let Λ_a be the kernel of ρ_a in K^{ac}. Then

$$\begin{aligned} Ker\, \mathfrak{f}_a\,(K^{ac}) &= \Lambda_a \oplus e_s(\rho_a^{-1}(D_{t'})) \\ &\simeq \Lambda_a \oplus \rho_a^{-1}(D_{t'})/D_t \\ &\simeq \Lambda_a \oplus D_{t'}/\rho_a(D_t) \\ &\simeq \Lambda_a \oplus D_{t'}/\rho_b(D_{t'}). \end{aligned}$$

Hence we get (iii).

For a Drinfeld module ϕ over K given by

$$\phi_T = T\tau^0 + g\tau + \Delta\tau^2,$$

the ***Hasse invariant*** of ϕ is defined to be the class of $g \ mod(K^*)^{q-1}$ for $j \neq 0$. For $j = 0$, $\Delta \ mod K^{*q^2-1}$ will be called the Hasse invariant of ϕ. Then the j-invariant and the Hasse invariant together determine a K-isomorphism class of Drinfeld modules over K as in the theory of elliptic curves. These notions are related with the Tate-Drinfeld modules in the following way (cf: [5], VIIIa);

THEOREM 1.2. *A Drinfeld module ϕ over K is K-isomorphic to a Tate-Drinfeld modue over K if and only if the followings are true;*

(i) $|\ j\ |> 1$
(ii) *The Hasse invariant is trivial.*

Proof. Assume that ϕ is K-isomorphic to a Tate-Drinfeld module ϕ^s with $|\ s\ |< 1$. Then (i) is trivial from the s-expansion of j. (ii) follows from the s-expansion of $\tilde{g}(s)$ and the trivial case of the Hensel's lemma for complete local ring for the equation $X^{q-1} - \tilde{g}(s) = 0$. Now assume (i) and (ii). By (i), $|\ \frac{1}{j}\ |< 1$, and so $|\ s\ |=|\ \frac{1}{j} + h(\frac{1}{j})\ |< 1$. Since $\tilde{g}(s) \in K^{*q-1}$, ϕ^t and ϕ have the same j-invariant and Hasse invariant, they are K-isomorphic.

REMARK. For a Drinfeld module ϕ over K with nonintegral j-invari ant, there exists a finite separable extension L of K so that ϕ is L-isomorphic to a Tate-Drinfeld module over K. In fact, we can take L to be the field $K(g^{\frac{1}{q-1}})$.

2. Drinfeld modules with non-Integral j-invariants

Let ϕ be a Drinfeld module with $|\ j\ |> 1$. Put $L = K(g^{\frac{1}{q-1}})$. We assume that every algebraic extension of K is contained in a fixed algebraic closure K^{ac} of K. Over L, ϕ is isomorphic to a Tate-Drinfeld module ϕ^s for some s in K, that is, there is an element ϵ in L such that

$$\phi^s = \epsilon \cdot \phi \cdot \epsilon^{-1}.$$

We shall call s the *period* of ϕ. Then it is clear tat $\epsilon^{q-1} \in K$. Let $\tilde{e}_s$ be the Tate-Drinfeld map of ϕ^s over L. Put

$$e_s = \epsilon^{-1} \cdot \tilde{e}_s.$$

Then we see easily that

$$\phi_a \circ e_s = e_s \circ \rho_a.$$

Let σ be an element of $Gal(L/K)$. Since $\tilde{e}_s$ is defined over K, we have

$$e_s(\sigma(x)) = \mu_\sigma \cdot \sigma(e_s(x))$$

where $\mu_\sigma = \frac{\sigma(\epsilon)}{\epsilon} \in \mathbb{F}_q^*$. Since $\epsilon^{1-q} = \tilde{g}(s)/g$ and $\tilde{g}(s) \in (K^*)^{q-1}$, we have

$$(*) \qquad \mu_\sigma = \sigma(g^{\frac{1}{q-1}})/g^{\frac{1}{q-1}}.$$

LEMMA 2.1. *Let ϕ' be a Drinfeld module with period s'. Suppose that $\rho_a(D_t) \subset D_{t'}$. Let u_a be the isogeny from ϕ to ϕ' associated to the isogeny $\mathfrak{f}_a$ from ϕ^s to $\phi^{s'}$. Then we have*

$$u_a(e_s(x)) = e_{s'}(\rho_a(x)).$$

Proo. This follows easily from the fact that $u_a \circ \epsilon^{-1} = \epsilon'^{-1} \circ \mathfrak{f}_a$ and $\mathfrak{f}_a \circ \tilde{e}_s = \tilde{e}_{s'} \circ \rho_a$.

Now we will prove the following Isogeny Theorem;

THEOREM 2.2. *Let ϕ be a Drinfeld module over K with non-integral j-invariant j. Let s be its period and g $mod_s (K^*)^{q-1}$ its Hasse invariant. Then a Drinfeld module ϕ' over K is K-isogenous to ϕ if and ony if its period s' is commensurable with s and its Hasse invariant is the same as that of ϕ.*

Proof. Let s (resp. s') be the period of ϕ (resp. ϕ') and t (resp. t') the $(q-1)$st root of s (resp. s'). Suppose first that the Hasse invariant of ϕ is trivial. Then ϕ is K-isomorphic to ϕ^s. Let u be the K-isogeny from ϕ' to ϕ. Since ϕ^s and $\phi^{s'}$ are isogenous, s and s' are commensurable. Let $\mathfrak{f}$ be the isogeny from $\phi^{s'}$ to ϕ^{s} associated to u. Then $\mathfrak{f}$ must be of the form $\mathfrak{f}_a$, so $\mathfrak{f}$ is defined over K. Now from the commutative diagram

$$\begin{array}{ccc} \phi' & \xrightarrow{u} & \phi \\ {\scriptstyle \epsilon'}\downarrow & & \downarrow{\scriptstyle \epsilon} \\ \phi^{s'} & \xrightarrow{\mathfrak{f}} & \phi^{s}, \end{array}$$

ϵ' must be an element of K because u, ϵ, and $\mathfrak{f}$ are defined over K. Therefore ϕ' has trivial Hasse invariant. The converse is trivial in this case.

Now assume that $g \mod (K^*)^{q-1}$ is not trivial. Let $L = K(g^{\frac{1}{q-1}})$ and u be a K-isogeny from ϕ' to ϕ. The periods do not depend on the extension field of K, s and s' must be commensurable. Viewing u as an L-isogeny, $g' \mod (L^*)^{q-1}$ is trivial by the previous discussion. Thus we can conclude that $K(g^{\frac{1}{q-1}}) = K(g'^{\frac{1}{q-1}})$. Hence $gg' \in (K^*)^{q-1}$ by Kummer theory. Conversely, assume that $\rho_a(t^{-1}) = \rho_b(t'^{-1})$ and $g \equiv g' \mod (K^*)^{q-1}$. Let u be the isogeny over L associated to the isogeny $\mathfrak{f}_a$ from ϕ^s to $\phi^{s'}$. Let $\mu_\sigma = \frac{\sigma(\epsilon)}{\epsilon}$ and $\mu'_\sigma = \frac{\sigma(\epsilon')}{\epsilon'}$ for $\sigma \in Gal(L/K)$. Then from (*), $\mu_\sigma = \mu'_\sigma$, since $g \equiv g' \mod (K^*)^{q-1}$. Since $e_s(\sigma(x)) = \mu_\sigma \cdot \sigma(e_s(x))$, we see that

$$\begin{aligned} u(\sigma(e_s(x))) &= u(\mu_\sigma^{-1} \cdot e_s(\sigma(x))) \\ &= \mu_\sigma^{-1} \cdot u(e_s(\sigma(x))) \\ &= \mu_\sigma^{-1} \cdot e_{s'}(\rho_a(\sigma(x))) \\ &= \mu_\sigma^{-1} \cdot e_{s'}(\sigma(\rho_a(x))) \\ &= \mu_\sigma^{-1} \cdot \mu_\sigma \cdot \sigma(e_{s'}(\rho_a(x))) \\ &= \sigma(u(e_s(x))). \end{aligned}$$

Hence $u \circ \sigma = \sigma \circ u$, which implies that u is defined over K.

3. Torsion Points

From the construction of $e_s(u)$, it is clear that $e_s(u)$ lies in $R[[u]]$. Let t^{-a} be any element ofK^{ac} such that $\rho_a(t^{-a}) = t^{-1}$ and λ_a a primitive a-th root of ρ. Put

$$\omega_{\alpha,\beta} = \rho_\alpha(\lambda_a) + \rho_\beta(t^{-a}), \qquad \alpha, \beta \in A/(a).$$

Then the a-torsion points of ϕ^t are

$$e_s(\omega_{\alpha,\beta}) = \phi^s_\alpha(e_s(\lambda_a)) + \phi^s_\beta(e_s(t^{-a})), \qquad \alpha, \beta \in A/(a).$$

Since $e_s(u)$ lies in $R[[u]]$, the nonintegral torsion points come from $\phi_\beta(e_s(t^{-a}))$. But $v(t) > 0$ implies that$v(t^{-a}) < 0$, and so

$$-v(t) = v(\rho_a(t^{-a})) = q^{deg\, a} v(t^{-a}).$$

Since $v(t)$ is fixed, $deg\, a$ must be bounded. Therefore one can conclude that the set

$$Tor_{\phi^s}(K)/Tor_{\phi^s}(R)$$

is finite.

A Drinfeld module ϕ over K is said to be *minimal* if ϕ is defined over R and $v(\Delta)$ is minimal among the Drinfeld modules over R which are K-isomorphic to ϕ. Let ϕ be a minimal Drinfeld module over K with nonintegral j-invariant j. From the remark at the end of the first section, we can find a finite separable extension L of K with the ring of integers S so that ϕ is L-isomorphic to a Tate-Drinfeld module ϕ^s over L, that is, there exists $c \in L$ such that

$$\phi^s = c \cdot \phi \cdot c^{-1}.$$

Since ϕ is minimal, $0 \leq v_L(c) < e$, where e is the ramification degree of L over K. Then the multiplication by c induces an injective homomorphism

$$K/R \xrightarrow{\cdot c} L/S.$$

Thus

$$Tor_\phi(K)/Tor_\phi(R) \xrightarrow{\cdot c} Tor_{\phi^s}(L)/Tor_{\phi^s}(S)$$

is injective. Since we know that $Tor_{\phi^s}(L)/Tor_{\phi^s}(S)$ is finite, w see that $Tor_\phi(K)/Tor_\phi(R)$ is finite. It is shown in [2] that if a minimal Drinfeld module ϕ has an integral j-invariant, then $Tor_\phi(K) = Tor_\phi(R)$. Hence we obtain the following analogue of the Theorem of Kodaira-Neron.

THEOREM 3.1. *For a minimal Drinfeld module ϕ of rank 2 over K, $Tor_\phi(K)/Tor_\phi(R)$ is finite.*

Now we can prove the following analogue of the criterion of Neron-Ogg- Shafarevich for a Drinfeld module to have good reduction, following the method in [7]. This theorem was proved by Takahash in [8] in a different way.

THEOREM 3.2. *Let ϕ be a minimal Drinfeld module. Let $\mathfrak{p}$ be an irreducible polynomial in A prime to the divisorial characteristic of the reduced Drinfeld module $\bar{\phi}$ over the residue field k. Then ϕ has good reduction over K if and only if $Tor_\phi[\mathfrak{p}^\infty]$ is unramified as a Galois module.*

Proof. The 'only if' part is trivial. Assume that $Tor_\phi[\mathfrak{p}^\infty]$ is unramified. Let K^{nr} be the maximal unramified extension of K. Choose n so that

$$q^{n\cdot deg\,\mathfrak{p}} > \sharp \mid Tor_\phi(K^{nr})/Tor_\phi(R^{nr}) \mid .$$

Since $Tor_\phi[\mathfrak{p}^n] \subset Tor_\phi(K^{nr})$, $Tor_\phi(K^{nr})$ contains a submodule isomorphic to $(A/\mathfrak{p}^n)^2$. But $Tor_\phi(K^{nr})/Tor_\phi(R^{nr})$ has order strictly less than $q^{n\cdot deg\,\mathfrak{p}}$, we see that $Tor_\phi(R^{nr})$ contains a submodule isomorphic to $(A/\mathfrak{p})^2$, from the exact sequence

$$0 \longrightarrow Tor_\phi(R^{nr}) \longrightarrow Tor_\phi(K^{nr}) \longrightarrow Tor_\phi(K^{nr})/Tor_\phi(R^{nr}) \longrightarrow 0.$$

We know that $Tor_\phi(\mathfrak{m}^{nr})$ has no nontrivial $\mathfrak{p}$-torsion ([2], Proposition 1.3). Thus $Tor_{\bar{\phi}}(k^{ac})$ must have a submodule isomorphic to $(A/\mathfrak{p})^2$, by the exact sequence

$$0 \longrightarrow Tor_\phi(\mathfrak{m}^{nr}) \longrightarrow Tor_\phi(R^{nr}) \longrightarrow Tor_{\bar{\phi}}(k^{ac}).$$

Now suppose that ϕ has bad reduction over K^{nr}. Then $\bar{\phi}$ has rank at most 1. Thus $Tor_{\bar{\phi}}(k^{ac})$ cannot contain a submodule of the form $(A/\mathfrak{p})^2$. Hence ϕ has good reduction over K^{nr}. Since K^{nr}/K is unramified, ϕ must have good reduction over K.

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