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ON THE CRITICAL MAPS OF THE DIRICHLET FUNCTIONAL WITH VOLUME CONSTRAINT

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1. Introduction

We consider a torus T, that is, a compact surface with genus 1 and $\Omega = D^2 \times S^1$ topologically with $\partial\Omega = T$, where D^2 is the open unit disk and S^1 is the unit circle. Let $w = (x,\ y)$ denote the generic point on T. For a smooth immersion $u : T \to R^3$, we define the Dirichlet functional by

$$E(u) = \frac{1}{2}\int_T |\nabla u|^2 dw$$

and the volume functional by

$$V(u) = \frac{1}{3}\int_T u \cdot u_x \wedge u_y dw.$$

Now we define a Sobolev subspace

$$W = \{u \in W^{1,2}(T,\ R^3) : V(u) = \frac{4\pi}{3}\}.$$

Then the Euler-Lagrange equation of E on the volume constrained set W is $\Delta u + \lambda u_x \wedge u_y = 0$, where λ is Lagrange multiplier. Let $\mathcal{M}$ be the set of all the maps $U : \Omega \to W$ such that

(1) $U(p) = U_p$ is continuous in p in the sense that

$$||U_p - U_q||_{1,2} = \Big(\int_T |\nabla U_p - \nabla U_q|^2 dw\Big)^{\frac{1}{2}} \to 0$$

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as $p \to q$ in Ω, and

(2) if $q \in T$ and $p \to q$, then U_p tends to a round sphere at q, that is, for p sufficiently close to $q \in T$, U_p maps a small neighborhood of q almost to a sphere while collapsing the complement of the neighborhood of q almost to a point.

For all $u \in W$ we have the following inequalities;

$$\frac{1}{2}\int_T |\nabla u|^2 dw \geq \int_T |u_x \wedge u_y| dw \geq (36\pi)^{\frac{1}{3}} V(u)^{\frac{2}{3}} = 4\pi.$$

The second inequality which we call the isoperimetric inequality becomes an equality only when u is a round sphere. So we obtain $E(u) = \frac{1}{2}\int_T |\nabla u|^2 dw > 4\pi$ for each $u \in W$. However, it is easy to see that $\inf_{u \in W} E(u) = 4\pi$. So a minimizing sequence in W cannot converge. Instead, we consider a minimax sequence which might converge to a map whose Dirichlet energy is strictly bigger than 4π.

In this paper we will show that

$$\inf_{U \in \mathcal{M}} \sup_{p \in \Omega} E(U_p) > 4\pi.$$

2. Main Theorems

LEMMA 1. *For each $U \in \mathcal{M}$, $\sup_{p \in \Omega} \int |\nabla U_p|^2 dw$ is achieved at some point $p = p(U) \in \Omega$.*

Proof. Since as $p \to q$ $U_p \to$ a round sphere at q, $\int_T |\nabla U_p|^2 dw \to 8\pi$ as $p \to q$. But for each $p \in \Omega$ we have $\int_T |\nabla U_p|^2 dw > 8\pi$. So the supremum is achieved at some point $p = p(U) \in \Omega$ and hence

$$\sup_{p \in \Omega} \int_T |\nabla U_p|^2 dw = \max_{p \in \Omega} \int_T |\nabla U_p|^2 dw = \int_T |\nabla U_{p(U)}|^2 dw.$$

The following theorem gives us an information about the way bubbles appear. Intuitively, bubbles occur where a sequence does not converge and we may have some other bubbles on each bubble, and so on. But the number of bubbles should be finite.

THEOREM 2. *Let $\{u^j\}$ be a bounded sequence in W satisfying $dE|_W(u^j) \to 0$ strongly in W^*, that is, $dE(u^j)+\lambda dV(u^j) \to 0$ strongly, where λ is Lagrange multiplier. Then there exist positive numbers $M_1 < \cdots < M_k$, a solution $u_0 \in W^{1,2}(T,\ R^3)$ of $\Delta u + \lambda u_x \wedge u_y = 0$ and solutions $u_l \in W^{1,2}(R^2,\ R^3)$ of $\Delta u + \lambda u_x \wedge u_y = 0$ in R^2 and $u_l(w) \to$ constant as $|w| \to \infty$ for $1 \le l \le M_k$ such that for a subsequence $j \to \infty$ $u^j \to u_0$ weakly in $W^{1,2}(T,\ R^3)$, the first few u_l's are obtained from $u^j - u_0$ in the following sense; for $1 \le l \le M_1 < M_k$*

$$u_l^j \equiv (u^j - u_0)|_{B_l}(r_l^j(\cdot - w_l^j)) \to u_l$$

weakly in $W^{1,2}(R^2,\ R^3)$ as $j \to \infty$, where B_l is a ball in R^2, $\{r_l^j\}$ and $\{w_l^j\}$ are sequences of radii and points in R^2, respectively, likewise, the next few u_m's, $M_1 < m \le M_2 < M_k$ are obtained as weak limits in $W^{1,2}(R^2,\ R^3)$ from $u_l^j - u_l$ for some $l = l(m)$, $1 \le l \le M_1 < M_k$, and so on, and finally the last few u_n's, $M_{k-1} < n \le M_k$, are obtained as strong limits in $W^{1,2}(R^2,\ R^3)$ from $u_i^j - u_i$ for some $i = i(n)$, $0 \le i \le M_k$ with $u_0^j \equiv u^j$. Moreover, we have

$$E(u^j) \to \sum_{l=0}^{M_k} E(u_l) \text{ and } V(u^j) \to \sum_{l=0}^{M_k} V(u_l).$$

For a proof of Theorem 2, see [1, 2].

COROLLARY 3. *If $\{u^j\}$ is a sequence in W satisfying $E(u^j) \to 4\pi$ and $dE|_W(u^j) \to 0$, then for j sufficiently large u^j is close to a round sphere at some point $w \in T$.*

Proof. By Theorem 2, we have solutions $u_0 \in W^{1,2}(T,\ R^3), u_1, \cdots,$ $u_k \in W^{1,2}(S^2,\ R^3)$ of $\Delta u + \lambda u_x \wedge u_y = 0$ such that

$$E(u^j) \to \sum_{i=0}^{k} E(u_i), \quad V(u^j) \to \sum_{i=0}^{k} V(u_i) \text{ as } j \to \infty.$$

By the hypothesis of $\{u^j\}$, $\sum_{i=0}^{k} E(u_i) = 4\pi$. But if $u_0 \in W^{1,2}(T,\ R^3)$ is not constant, then $E(u_0) > 4\pi$. So $E(u_0) = V(u_0) = 0$ and hence

$$\sum_{i=1}^{k} E(u_i) = 4\pi, \quad \sum_{i=1}^{k} V(u_i) = \frac{4\pi}{3}.$$

Since $\Delta u_i + \lambda u_{ix} \wedge u_{iy} = 0$ for each $1 \le i \le k$, we obtain

$$-\sum_{i=1}^{k} \int_{S^2} |\nabla u_i|^2 dw + \lambda \sum_{i=1}^{k} \int_{S^2} u_i \cdot u_{ix} \wedge u_{iy} dw = 0.$$

So we have $-8\pi + 4\pi\lambda = 0$, that is, $\lambda = 2$. By Brezis and Coron's result (see [3]), each u_i, $1 \le i \le k$, is a conformal branched covering of a sphere with radius $\frac{2}{\lambda} = 1$. Note that $E(u_i) = \text{Area}(u_i)$, $1 \le i \le k$. So there exists only one bubble u_1 at some point $w \in T$. Hence u^j is close to u_1 for sufficiently large j.

LEMMA 4. (Ekeland's Variational Principle) *Let $\{u^j\}$ be a sequence in W such that*

$$E(u^j) \le \inf_{u \in W} E(u) + \frac{1}{j^2}.$$

Then there exists a sequence $\{v^j\}$ in W such that $E(v^j) \to \inf_{u \in W} E(u)$, $dE|_W(v^j) \to 0$ and $||u^j - v^j||_{1,2} < \frac{1}{j}$.

For a proof of Lemma 4, see [4].

THEOREM 5. *Let $s = \inf_{U \in \mathcal{M}} \max_{p \in \Omega} \frac{1}{2} \int_T |\nabla U_p|^2 dw$. Then $s > 4\pi$.*

Proof. Suppose that $s = 4\pi$. Then we may choose a sequence $\{U^j\}$ in $\mathcal{M}$ such that

$$\max_{p \in \Omega} \int_T |\nabla U_p^j|^2 dw \le 8\pi + \frac{1}{j^2}.$$

So for each $p \in \Omega$, there exist $u_p^j \in W$ such that

$$||U_p^j - u_p^j||_{1,2} < \frac{1}{j}, \quad E(u_p^j) \to 4\pi \text{ and } \ dE|_W(u_p^j) \to 0$$

by Lemma 4. If we apply Corollary 3 with u_p^j, we can conclude that for sufficiently large j u_p^j is close to a bubble at $w_p \in T$ and so is U_p^j.

We claim that $p \mapsto w_p : \Omega \to T$ is continuous. Let $\varepsilon > 0$. Since $U^j : \Omega \to W$ is continuous and $||U_p^j - u_p^j||_{1,2} \to 0$ as $j \to \infty$, there exist $\delta = \delta(\varepsilon) > 0$ and $J > 0$ such that $||U_p^j - U_q^j||_{1,2} < \varepsilon$ whenever $\text{dist}(p, q) < \delta$, and $||u_p^j - U_p^j||_{1,2} < \varepsilon$,

$$||u_q^j - U_q^j||_{1,2} < \varepsilon \quad \text{for } j \ge J.$$

So we have

$$||u_p^j - u_q^j||_{1,2} \le ||u_p^j - U_p^j||_{1,2} + ||U_p^j - U_q^j||_{1,2} + ||U_q^j - u_q^j||_{1,2} < 3\varepsilon,$$

for $p,\ q \in \Omega$ with $\operatorname{dist}(p,\ q) < \delta$ and $j \ge J$.

Since w_p and w_q are the unique points in T where subsequences $u_p^j - u_{p_0}$ and $u_q^j - u_{q_0}$ do not converge strongly to 0 in $W^{1,2}(T,\ R^3)$, we have for some $\nu > 0$

$$\lim_{\rho \to 0} \liminf_{j \to \infty} \int_{B_\rho(w_p)} |\nabla u_p^j|^2 dw \ge \nu^2$$

and

$$\lim_{\rho \to \infty} \liminf_{j \to \infty} \int_{B_\rho(w_q)} |\nabla u_q^j|^2 dw \ge \nu^2.$$

Thus if $\operatorname{dist}(p,\ q) < \delta = \delta(\varepsilon)$, then for any $\rho > 0$

$$\begin{aligned}\nu &\le \liminf_{j \to \infty} \Big(\int_{B_\rho(w_p)} |\nabla u_p^j|^2 dw\Big)^{\frac{1}{2}} \\ &\le \liminf_{j \to \infty} \Big(\Big(\int_{B_\rho(w_p)} |\nabla u_p^j - \nabla u_q^j|^2 dw\Big)^{\frac{1}{2}} + \Big(\int_{B_\rho(w_p)} |\nabla u_q^j|^2 dw\Big)^{\frac{1}{2}}\Big) \\ &< 3\varepsilon + \liminf_{j \to \infty} \Big(\int_{B_\rho(w_p)} |\nabla u_q^j|^2 dw\Big)^{\frac{1}{2}}.\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain $p \to q$ and

$$\liminf_{j \to \infty} \int_{B_\rho(w_p)} |\nabla u_q^j|^2 dw \ge \nu^2 \quad \text{for all } \rho > 0.$$

Hence $w_p \to w_q$ in T which is a compact set.

Now we define $T \xrightarrow{i} \Omega \cup T \xrightarrow{f} T$, where i is an inclusion map, $f(p) = w_p$ for $p \in \Omega$ and $f(q) = q$ for $q \in T$ so that $f \circ i$ is the identity map on T. Since for $U \in \mathcal{M}$, U is continuous and $U_p \to$ a round sphere at $q \in T$ as $p \in \Omega \to q$, f is continuously defined. In fact, for $p \in \Omega$ close to $q \in T$, there exists a sequence $\{U_p^j\}$ such that U_p^J is close to a bubble at $w_p \in T$ for large J. But by the way we have defined U^J, U_p^J is already close to a bubble at $q \in T$. So w_p is close to q. So we have $\pi_1(T) \xrightarrow{i_*} \pi_1(\Omega \cup T) \xrightarrow{f_*} \pi_1(T)$ and $f_* \circ i_*$ is an isomorphism. But $\pi_1(T) = Z \times Z$ and $\pi_1(\Omega \cup T) = Z$. Hence $Z \times Z \to Z \to Z \times Z$ is an isomorphism, which is impossible. So $s > 4\pi$.

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