Bull. Korean Math. Soc. **32** (1995), No. 2, pp. 287–296

STABILITY OF THE MULTIPLE OBJECTIVE LINEAR STOCHASTIC PROGRAMMING PROBLEMS

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1. Introduction

Wets ([4],[5],[6]) considered single objective linear two-stage programming problem under uncertainty with complete recourse. Artstein, Dupačova, Römisch, Schultz and Wets studied stability of this problem in depth. But in many real world problems to make best decision, we need multiple objective functions. So we consider the following multiple objective two-stage programming problems with complete fixed recourse.

$$
\begin{aligned}
(1) \qquad \text{VMIN} \quad & g(x) + E_\xi[\min \bar{q}'y] \cdot \mathbf{1} \\
\text{subject to} \quad & Ax = b \\
& Dx + Wy = \xi, \quad \xi \text{ on } (\Xi, \Im, \mu) \\
& x \geq 0, \quad y \geq 0,
\end{aligned}
$$

where A is an $m \times n$ matrix, D is an $\bar{m} \times n$ matrix, x is an $n-$vector, $\bar{q}$ and y are $\bar{n}-$vectors, $\mathbf{1} = (1, \cdots, 1)'$ is a $r-$vector, b is an $m-$vector and ξ is a random vector defined on the probability space $(\Xi, \Im, \mu)$, $g(x) = (g_1(x), \cdots, g_r(x))'$, $g_i(x) = c_{i1}x_1 + \cdots + c_{in}x_n$, $1 \leq i \leq r$, is a linear function, W is an $\bar{m} \times \bar{n}$ matrix such that $\{Wy : y \in \mathbf{R}^{\bar{n}}, \ y \geq 0\} = \mathbf{R}^{\bar{m}}$ and $\{u \in \mathbf{R}^{\bar{m}} \mid W'u \leq \bar{q}\} \neq \emptyset$, and E_ξ denotes the mean operator with respect to ξ.

All quantities considered here belong to the reals. We use the notation ξ to denote a random vector of dimension $\bar{m}$, as well as the specific values assumed by this random variable.

We approach problem (1) by using the weight parameter. Since the weight parameter depends on decision makers it is important to study

Received April 7, 1994.

the stability of problem (1) with respect to this parameter. So we want to observe the stability of problem (1) with respect to the weight parameter and probability distributions of the random vector, respectively. We are going to use D. Klatte's parametric programming analysis([2]). Let us consider the following programming problem with a parameter $t \in T$:

$$(2) \qquad p(t) \, : \, \min\{f(x,t) : x \in M(t)\}$$

where T is a metric space with distance function $d(\cdot,\cdot)$, M is a closed-valued multifunction from T into $\mathbf{R}^n$, and $f : \mathbf{R}^n \times T \to \mathbf{R}$ is a continuous function. Given $Q \subset \mathbf{R}^n$, for any $t \in T$, we define

$$M_Q(t) = M(t) \cap clQ,$$

$$\rho_Q(t) = \inf\{f(x,t) \mid x \in M_Q(t)\},$$

$$\psi_Q(t) = \{x \in M_Q(t) \mid f(x,t) = \rho_Q(t)\},$$

where clQ denotes the closure of Q. We call ρ_Q the optimal value function with respect to clQ and ψ_Q the optimal set function with respect to clQ. Now we shall give some basic definitions and theorems.

DEFINITION 1.1. Given $t^o \in T$, a nonempty set $X \subset \mathbf{R}^n$ is called a *complete local minimizing set(CLM)* for $f(\cdot, t^o)$ on $M(t^o)$ if there is an open set Q containing X such that $X = \psi_Q(t^o)$.

Note that a CLM set with respect to $p(t^o)$ is always a subset of the set of all local minimizers of $p(t^o)$ and CLM set are closed under our general assumptions on the problem $p(\cdot)$.

DEFINITION 1.2. $M : T \to \mathbf{R}^n$, is said to be *closed* at t^o if and only if $t^{(k)} \to t^o$, $x^{(k)} \to x^o$, as $k \to \infty$, and $x^{(k)} \in M(t^{(k)}) \Rightarrow x^o \in M(t^o)$.

DEFINITION 1.3. A multifunction M from T to $\mathbf{R}^n$ is said to be *pseudo-Lipschitzian* at (x^o, t^o), where $t^o \in T$ and $x^o \in M(t^o)$, if there are neighborhoods $U = U(t^o)$ and $V = V(x^o)$ and a positive real number L such that both

$$M(t) \cap V \subset M(t^o) + Ld(t,t^o)B_n$$

and

$$M(t^o) \cap V \subset M(t) + Ld(t, t^o)B_n,$$

hold for all $t \in U$, where B_n is the closed unit ball in $\mathbf{R}^n$, and

$$X + \gamma B_n = \{x + \gamma u \mid x \in X,\ u \in B_n\},$$

for $X \subset \mathbf{R}^n$ and $\gamma \in \mathbf{R}$.

THEOREM 1.4. ([2]) *Consider the parametric program $p(t)$: fix some $t^o \in T$ with the following conditions:*

$(C1)$ Assume that there exists a bounded open subset V of $\mathbf{R}^n$ and a nonempty subset X of V such that $X = \psi_V(t^o)$.

$(C2)$ Let the multifunction M be closed-valued and closed at t^o.

$(C3)$ Let M be a pseudo-Lipschitzian at each pair $(x^o, t^o) \in \psi_V(t^o) \times \{t^o\}$.

$(C4)$ Suppose there are real numbers $p \in (0, 1]$, $L_f > 0$ and $\delta_f > 0$ such that

$$| f(x, t^o) - f(y, t) | \leq L_f(\| x - y \| + d(t, t^o)^p)$$

for each x, $y \in clV$ and each t satisfying $d(t, t^o) < \delta_f$.

Then the following conclusions hold:

(a) *The multifunction ψ_V is upper semicontinuous at t^o, i.e., for each $\epsilon > 0$ there exists $\eta > 0$ such that*

$$\psi_V(t) \subset \psi_V(t^o) + \epsilon B_n \quad \text{when } d(t, t^o) < \eta.$$

(b) *There exist positive real numbers δ_ρ* and *L_ρ such that $\psi_V(t) \neq \emptyset$ is a CLM set for $f(\cdot, t)$ on $M(t)$ and such that*

$$| \rho_V(t) - \rho_V(t^o) | \leq L_\rho d(t, t^o)^p \quad \text{whenever } d(t, t^o) < \delta_\rho.$$

2. Approach to multiple objective stochastic programming problem

When we find the efficient solutions or properly efficient solutions of the multiple objective programming problem we usually convert the problem to single objective programming problem by using the various ways. We will convert problem (1) to single objective programming problem by using the weight parameter.

It is clear that we could also write problem (1) as:

$$\begin{aligned} \text{VMIN} \quad & g(x) + E_\xi[\min \bar{q}'y : Wy = \xi - Dx,\ y \geq 0] \cdot \mathbf{1} \\ \text{subject to} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

Now we define a feasible solution to problem (1).

DEFINITION 2.1. A *feasible solution* to (1) is a vector x such that it satisfies the first stage constraints and such that for any $\xi \in \Xi$, it is always possible to find a feasible solution to the second stage problem $\min\{\bar{q}'y \mid Wy = \xi - Dx,\ y \geq 0\}$.

Let K be the set of feasible solutions of (1). Then K is a convex polyhedron. Define

$$Q(x, \xi) = \min\{\bar{q}'y \mid Wy = \xi - Dx,\ y \geq 0\}$$

and

$$Q(x) = E_\xi[Q(x, \xi)].$$

Then $Q(x)$ is convex and continuous. So we have an equivalent programming problem to (1).

$$\begin{aligned} \text{VMIN} \quad & F(x) = (g_1(x) + Q(x), \cdots, g_r(x) + Q(x))' \\ \text{subject to} \quad & x \in K \end{aligned} \tag{3}$$

DEFINITION 2.2. The vector x^* is an *efficient solution* of

$$\begin{aligned} \text{VMIN} \quad & F(x) = (f_1(x), \cdots, f_r(x))' \\ \text{subject to} \quad & x \in K \end{aligned}$$

if and only if there exists no $x \in K$ such that $f_i(x) \leq f_i(x^*)$ for $i = 1, \cdots, r$ and such that for at least one i_o one has $f_{io}(x) < f_{io}(x^*)$.

DEFINITION 2.3. The vector x^* is a *properly efficient solution* of

$$\begin{aligned} \text{VMIN} \quad & F(x) = (f_1(x), \cdots, f_r(x))' \\ \text{subject to} \quad & x \in K \end{aligned}$$

if and only if it is efficient and if there exists a scalar $\bar{M} > 0$ such that for each i and each $x \in K$ satisfying $f_i(x) < f_i(x^*)$, there exists at least one j such that $f_j(x^*) < f_j(x)$ and $(f_i(x^*) - f_i(x))/(f_j(x) - f_j(x^*)) \leq \bar{M}$.

THEOREM 2.4. ([1]) *Let K be a convex set, and let the f_i be concave on K. Then x^o is properly efficient in*

$$\text{VMAX} \quad F(x) = (f_1(x), \cdots, f_r(x))' \quad \text{subject to } x \in K$$

if and only if x^o is optimal in

$$\max \sum_{i=1}^{r} \lambda_i f_i(x) \text{ subject to } x \in K$$

for some λ with strictly positive components with $\sum_{i=1}^{r} \lambda_i = 1$.

Consider the following scalar minimization problem:

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{r} \lambda_i g_i(x) + Q(x) \\ \text{subject to} \quad & x \in K, \end{aligned} \tag{4}$$

where $\sum_{i=1}^{r} \lambda_i = 1,\ \lambda_i > 0,\ i = 1, \cdots, r$.

COROLLALLY 2.5. *x^* is properly efficient in problem (3) if and only if x^* is optimal in problem (4) for some λ with strictly positive components.*

Proof. It is clear.

Thus from this Cororally 2.5 we can find all the properly efficient solutions to problem (3) by searching all λ such that $\lambda_i > 0$ and $\sum_{i=1}^{r} \lambda_i = 1$.

3. Stability analysis

In the treatment of stochastic programming problems, their stability with respect to perturbation of the distributions of the underlying random variables plays an essential role. Since the weight parameter depends on decision makers, the stability with respect to weight parameter is also important. We investigate the stability of the optimal solution set and optimal value functions to problem (4) with respect to weight parameter λ and probability measure μ, respectively. We define a suitable metric on the weight parameter space and probability measure space. We define the bounded Lipschitz metric β on $\wp(\mathbf{R}^{\bar{m}})$ as follows:

$$\beta(\mu, \nu) =$$

$$\sup\Big\{ \Big| \int_{\mathbf{R}^{\bar{m}}} g(\xi)\mu(d\xi) - \int_{\mathbf{R}^{\bar{m}}} g(\xi)\nu(d\xi) \Big| : \; g : \mathbf{R}^{\bar{m}} \to \mathbf{R}, \; \| g \|_{BL} \leq 1 \Big\},$$

for any μ and $\nu \in \wp(\mathbf{R}^{\bar{m}})$, where

$$\| g \|_{BL} = \sup_{\xi \in \mathbf{R}^{\bar{m}}} | g(\xi) | + \sup_{\xi \neq \tilde{\xi}} \frac{| g(\xi) - g(\tilde{\xi}) |}{d(\xi, \tilde{\xi})} < \infty,$$

and

$$d(\xi, \tilde{\xi}) = \sqrt{(\xi_1 - \tilde{\xi}_1)^2 + \cdots + (\xi_{\bar{m}} - \tilde{\xi}_{\bar{m}})^2},$$

for $\xi = (\xi_1, \cdots, \xi_{\bar{m}})$, $\tilde{\xi} = (\tilde{\xi}_1, \cdots, \tilde{\xi}_{\bar{m}})$.

Note that the metric β metrizes the topology of weak convergence on $\wp(\mathbf{R}^{\bar{m}})$.

Define a metric d_1 on Λ by

$$d_1(\lambda, \lambda_o) = \sqrt{(\lambda_1 - \lambda_{o1})^2 + (\lambda_2 - \lambda_{o2})^2 + \cdots + (\lambda_r - \lambda_{or})^2},$$

for any $\lambda,\ \lambda_o \in \Lambda$.

Then (Λ, d_1) is a metric space and $(\wp(\mathbf{R}^{\bar{m}}; q, k), \beta)$ is also a metric space.

Let $q \in (1, \infty)$ and $k \in (0, \infty)$. Define

$$\wp(\mathbf{R}^{\bar{m}}; q, k) = \{\mu \in \wp(\mathbf{R}^{\bar{m}}) \mid \int_{\mathbf{R}^{\bar{m}}} \| \xi \|^{2q} \mu(d\xi) \leq k\},$$

where $\wp(\mathbf{R}^{\bar{m}})$ is the set of all Borel probability measures on $\mathbf{R}^{\bar{m}}$. Denote the objective function by

$$F(x, \lambda, \mu) = \sum_{i=1}^{r} \lambda_i g_i(x) + \int_{\mathbf{R}^{\bar{m}}} Q(x, \xi) \mu(d\xi).$$

Define a feasible solution set K by

$$K = \{x \mid Ax = b, x \geq 0\} \cap \{x \mid \forall \xi,\ \exists\ y \geq 0 \text{ such that } Wy = \xi - Dx\}.$$

Also, define an optimal value function $\varphi: \Lambda \times \wp(\mathbf{R}^{\bar{m}}; q, k) \longrightarrow \mathbf{R}$ and an optimal set function $\psi: \Lambda \times \wp(\mathbf{R}^{\bar{m}}; q, k) \longrightarrow K$ by

$$\varphi(\lambda, \mu) = \inf_{x \in K} F(x, \lambda, \mu),$$

$$\psi(\lambda, \mu) = \{x \in K \mid F(x, \lambda, \mu) = \varphi(\lambda, \mu)\}.$$

Since for any fixed parameter (λ, μ) problem (4) is a convex programming problem, local optimal solutions and local optimal values are global optimal solutions and global optimal values, respectively. Fixing $\lambda = \lambda_o$, and regarding μ as a parameter we have the following result.

THEOREM 3.1. *Fix $(\lambda_o, \mu_o) \in \Lambda \times \wp(\mathbf{R}^{\bar{m}}; q, k)$ and assume that $\psi(\lambda_o, \mu_o)$ is nonempty and bounded. Then it follows that:*

(a) *For any $\varepsilon_1 > 0$ there exists $\eta_1 > 0$ such that*

$$\psi(\lambda_o, \mu) \subset \psi(\lambda_o, \mu_o) + \varepsilon_1 B_n,$$

whenever $(\lambda_o, \mu) \in \Lambda \times \wp(\mathbf{R}^{\bar{m}}; q, k)$ and $\beta(\mu, \mu_o) < \eta_1$.

(b) *There exist positive reals $\delta_{\rho,1}$, $L_{\rho,1}$ such that $\psi(\lambda_o, \mu) \neq \emptyset$ and*

$$| \rho(\lambda_o, \mu) - \rho(\lambda_o, \mu_o) | \leq L_{\rho,1} \beta(\mu, \mu_o)^{1-\frac{1}{q}},$$

whenever $\mu \in \wp(\mathbf{R}^{\bar{m}}; q, k)$ and $\beta(\mu, \mu_o) < \delta_{\rho,1}$.

Proof. Take $(T, d) = (\wp(\mathbf{R}^{\bar{m}}; q, k), \beta)$, a metric space. We check the conditions of Theorem 1.4.

(C_1) Since $\psi(\lambda_o, \mu_o)$ is a global minimizing solution set and bounded by assumption, $\psi(\lambda_o, \mu_o)$ is a bounded complete local minimizing set.

(C_2) and (C_3) are clear.

(C_4) Let $B \subset \mathbf{R}^{\bar{n}}$ be a nonempty and compact set.

$$\begin{aligned}
&| F(x, \lambda_o, \mu_o) - F(y, \lambda_o, \mu) | \\
&= | \sum_{i=1}^{r} \lambda_{oi} g_i(x) + \int_{\mathbf{R}^{\bar{m}}} Q(x, \xi) \mu_o(d\xi) \\
&\quad - \sum_{i=1}^{r} \lambda_{oi} g_i(y) - \int_{\mathbf{R}^{\bar{m}}} Q(y, \xi) \mu(d\xi) | \\
&\leq | \sum_{i=1}^{r} \lambda_{oi} g_i(x) - \sum_{i=1}^{r} \lambda_{oi} g_i(y) | \\
&\quad + | \int_{\mathbf{R}^{\bar{m}}} Q(x, \xi) \mu_o(d\xi) - \int_{\mathbf{R}^{\bar{m}}} Q(y, \xi) \mu(d\xi) |
\end{aligned}$$

Let $\bar{c} = \max\{\| c_1 \|, \cdots, \| c_r \|\}$, where c_i is a coefficient matrix of i-th objective function, $i = 1, \cdots, r$. Then first part of right hand side of inequality is bounded by $\bar{c} \| x - y \|$. Since $Q(x, \cdot)$ is convex in x, $Q(x, \cdot)$

is Lipschitzian on B. Therefore there exists some constant $\bar{b} > 0$ such that

$$| \int_{\mathbf{R}^m} (Q(x,\xi) - Q(y,\xi))\,\mu_o d\xi \,| \leq \bar{b} \parallel x - y \parallel .$$

For the recourse part it is not hard to verify that for $x \in B$, there exists $M > 0$ such that

$$\begin{aligned} & | \, Q(x,\xi) - Q(x,\tilde{\xi}) \, | \\ \leq & M \max\{\parallel \bar{q} \parallel + \parallel \xi - Dx \parallel, \parallel \bar{q} \parallel + \parallel \tilde{\xi} - Dx \parallel\} \parallel \xi - \tilde{\xi} \parallel \\ \leq & MK \max\{\parallel \bar{q} \parallel + \parallel \xi \parallel + \parallel D \parallel, \parallel \bar{q} \parallel + \parallel \tilde{\xi} \parallel + \parallel D \parallel\} \parallel \xi - \tilde{\xi} \parallel, \end{aligned}$$

where $K = \max\{1, \bar{p}\}$, and $\bar{p} = \max\{\parallel x \parallel | \, x \in B\}$.

Taking $L(t) = MK(\parallel \bar{q} \parallel + \parallel D \parallel) + MKt$, there exists $C > 0$ such that for $q > 1$, we have

$$\begin{aligned} & | \int_{\mathbf{R}^m} Q(y,\xi)\,\mu_o(d\xi) - \int_{\mathbf{R}^m} Q(y,\xi)\,\mu(d\xi) \, | \\ \leq & C(1 + M_q(\mu) + M_q(\mu_o))\beta(\mu,\mu_o)^{1-\frac{1}{q}}, \end{aligned}$$

where $M_q(\mu) = (\int_{\mathbf{R}^m} L_1(\parallel \xi \parallel)^q \, \mu(d\xi))^{\frac{1}{q}}, \;\; L_1(t) = L(t) \cdot t$.

Therefore

$$| \, F(x,\lambda_o,\mu_o) - F(y,\lambda_o,\mu) \, | \leq L_{\rho,1}(\parallel x - y \parallel + \beta(\mu,\mu_o)^{1-1/q}),$$

where

$$L_{\rho,1} = \max\{\bar{c} + \bar{b}, \; C(1 + M_q(\mu) + M_q(\mu_o))\}.$$

Now fixing $\mu = \mu_o$ and regarding λ as only a parameter we have the following result.

THEOREM 3.2. *Fix $(\lambda_o, \mu_o) \in \Lambda \times \wp(\mathbf{R}^{\bar{m}}; q, k)$ and assume that $\psi(\lambda_o, \mu_o)$ is nonempty and bounded. Then it follows that:*

(a) *For any $\varepsilon_2 > 0$, there exists $\eta_2 > 0$ such that*

$$\psi(\lambda,\mu_o) \subset \psi(\lambda_o,\mu_o) + \varepsilon_2 B_n \;\; if \; d_1(\lambda,\lambda_o) < \eta_2.$$

(b) *There exist positive reals* $\delta_{\rho,2}, L_{\rho,2}$ *such that* $\psi(\lambda, \mu_o) \neq \emptyset$ *and*

$$| \rho(\lambda, \mu_o) - \rho(\lambda_o, \mu_o) | \leq L_{\rho,2} d_1(\lambda, \lambda_o) \;\; if \;\; d_1(\lambda, \lambda_o) < \delta_{\rho,2}.$$

Proof. Define (T, d) as (Λ, d_1). We check the conditions of Theorem 1.4. Conditions $(C_1), (C_2)$, and (C_3) are clear.

Let us verify (C_4).

$$\begin{aligned} &| F(x, \lambda_o, \mu_o) - F(y, \lambda, \mu_o) | \\ \leq &| \sum_{i=1}^{r} \lambda_{oi} g_i(x) - \sum_{i=1}^{r} \lambda_i g_i(y) | \\ &+ | \int_{\mathbf{R}^m} Q(x, \xi) \mu_o(d\xi) - \int_{\mathbf{R}^m} Q(y, \xi) \mu_o(d\xi) | \\ \leq &\bar{c} \| x - y \| + r \, \bar{c} \bar{p} \| \lambda_o - \lambda \| + \bar{b} \| x - y \|, \\ \leq &L_{\rho,2} \{ \| x - y \| + \| \lambda_o - \lambda \| \}, \end{aligned}$$

where $\bar{c} = \max\{\| c_1 \|, \ , \cdots, \| c_r \|\}$ and $L_{\rho,2} = \max\{\bar{c} + \bar{b}, r\bar{c}\bar{p}\}$, and $\bar{p} = \max\{\| x \| | \ x \in B\}$.

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