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NATURALLY REDUCTIVE HOMOGENEOUS SPACE $R^4\backslash(0)$

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1. Introduction and main results

The complete classification of the simply connected four dimensional naturally reductive homogeneous spaces is given in [3].

In this paper, we treat a '*not simply connected*', reductive homogeneous space $(R^\times SU(2), g)$, or $(R^4\backslash(0), g)$, with a left invariant Riemannian metric g whose hypersurface is $(SU(2), \bar{g})$, or $(S^3, \bar{g})$, where $\bar{g}$ is an arbitrary left invariant Riemannian metric on $SU(2)$. We get a necessary and sufficient condition in order for such a reductive homogeneous space $(R^\times SU(2), g)$ to be naturally reductive. And, we show that scalar curvature of $(R^\times SU(2), g)$ and the above condition are related subjects. Moreover, we obtain a necessary and sufficient condition in order for an almost complex manifold $(R^\times SU(2), g, J)$ to be a complex manifold.

Now, we introduce the Sugahara Lemma:

LEMMA A (CF. [5]). *Let $\bar{g}$ be a left invariant Riemannian metric on $SU(2)$. Let $< \, , \, >$ be an inner product on $\mathfrak{su}(2)$ defined by $< X, Y >:= \bar{g}_e(X_e, Y_e)$, where $\mathfrak{su}(2)$ is the Lie algebra of $SU(2)$, $X, Y \in \mathfrak{su}(2)$, and e is the identity matrix of $SU(2)$. Then there exists an orthonormal basis (V_1, V_2, V_3) of $\mathfrak{su}(2)$ with respect to $< \, , \, >_o$ such that*

$$(1.1) \qquad \begin{cases} [V_1, V_2] = (1/\sqrt{2})V_3, & [V_2, V_3] = (1/\sqrt{2})V_1, \\ [V_3, V_1] = (1/\sqrt{2})V_2, & < V_i, V_j >= \delta_{ij}\, a_i^{\,2}, \end{cases}$$

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where $<\ ,\ >_o$ *is an inner product induced from the Killing form of* $\mathfrak{su}(2)$ *and* $a_i, (i = 1, 2, 3)$, *are positive constant real numbers determined by the given left invariant Riemannian metric* $\bar{g}$ *of* $SU(2)$.

For a given left invariant Riemannian metric $\bar{g}$ on $SU(2)$, we fix an orthonormal basis (V_1, V_2, V_3) of $\mathfrak{su}(2)$ with respect to $<\ ,\ >_o$ with the property (1.1) in Lemma A and denote by $\bar{g}(a_1, a_2, a_3)$, or simply by $\bar{g}(a)$, the left invariant Riemannian metric on $SU(2)$ which is determined by positive real numbers a_1, a_2, a_3 in Lemma A.

In this paper, we get under the notations of Lemma A

THEOREM 1. *Let* $\bar{g}(a)$ *be a left invariant Riemannian metric on* $SU(2)$. *Let* $g(a)$ *be a left invariant Riemannian metric on* $R^\times SU(2)(=: G)$ *determined by an orthonormal basis* $\{X_1 := V_1/a_1, X_2 := V_2/a_2, X_3 := V_3/a_3, X_4 := I_2/a_4\}$ *on* $T_e(G)$, *where* a_4 *is a positive constant and* I_2 *the unit matrix of order 2. Then,* $(G, g(a))$ *is naturally reductive if and only if* $a_1 = a_2 = a_3$.

COROLLARY 2. *Assume* $a_2 = a_3$ *in* $(G, g(a))$ *of* Theorem 1. *Then,* $(G, g(a))$ *is naturally reductive if and only if the scalar curvature of* $(G, g(a))$ *is* $3/4{a_2}^2$.

THEOREM 3. *Under the assumption of* Theorem 1, *let* J *be a tensor field of type* (1.1) *on* G *defined by* $JX_1 = X_4$, $JX_2 = X_3$, $JX_3 = -X_2$ *and* $JX_4 = -X_1$. *Then, in order for the structure tensor field* J *of an almost complex manifold* $(G, g(a), J)$ *to be a complex structure, it is necessary and sufficient that* $a_2 = a_3$.

COROLLARY 4. *If* $(G, g(a))$ *is a naturally reductive Riemannian homogeneous space, then* $(G, g(a), J)$ *in* Theorem 3 *is a complex manifold.*

2. Preliminaries

2.1. A homogeneous space K/T of a connected Lie group K is called reductive if the following condition is satisfied: in the Lie algebra $\mathfrak{k}$ of K there exists a subspace $\mathfrak{m}$ such that $\mathfrak{k} = \mathfrak{m} + \mathfrak{t}$ (direct sum of vecter spaces) and $Ad(t)\mathfrak{m} \subset \mathfrak{m}$ for all $t \in T$, where $\mathfrak{t}$ is the subalgebra of $\mathfrak{k}$ corresponding to the identity component T_o of T and $Ad(t)$ denotes the adjoint representation of T in $\mathfrak{k}$.

Let $<\ ,\ >$ be an inner product which is invariant with respect to $Ad(T)$ on $\mathfrak{m}$. This inner product $<\ ,\ >$ determines an invariant Riemannian metric g on K/T. Then the connection function α (cf. [1, p. 43]) on $\mathfrak{m}\times\mathfrak{m}$ corresponding to the K-invariant Riemannian connection of a reductive Riemannian homogeneous space $(K/T,g)$ is given as follows (cf. [1, p. 52]):

$$\alpha(X,Y) = (1/2)\,[X,Y]_{\mathfrak{m}} + U(X,Y), \qquad (X,Y\in\mathfrak{m}), \tag{2.1}$$

where $U(X,Y)$ is determined by

$$\begin{aligned} &2<U(X,Y),Z> \\ &= <[Z,X]_{\mathfrak{m}},Y> + <X,[Z,Y]_{\mathfrak{m}}>, \qquad (X,Y,Z\in\mathfrak{m}), \end{aligned} \tag{2.2}$$

where $X_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of an element $X\in\mathfrak{k}=\mathfrak{t}+\mathfrak{m}$. The curvature tensor R at $\{T\}\in K/T$ is given by

(2.3)
$$\begin{aligned} R(X,Y)Z =&\alpha(X,\alpha(Y,Z))-\alpha(Y,\alpha(X,Z)) \\ &-\alpha([X,Y]_{\mathfrak{m}},Z)-[[X,Y]_{\mathfrak{t}},Z], \qquad (X,Y,Z\in\mathfrak{m}). \end{aligned}$$

A reductive Riemannian homogeneous space $(K/T,g)$ is called naturally reductive if the following condition is satisfied : Let $x(s)$ be the 1-parameter subgroup of K generated by an arbitrary given element $X\in\mathfrak{m}$, and let $x^*(s)$ be the image of $x(s)$ by the projection π of K onto K/T. Then $x^*(s)$ is a geodesic in $(K/T,g)$. A reductive Riemannian homogeneous space $(K/T,g)$ is naturally reductive if and only if

$$U(X,Y)=0, \quad (X,Y\in\mathfrak{m}). \tag{2.4}$$

2.2. Let a Riemannian manifold (M^{2n},g) have a tensor field J of type (1,1) such that $J^2=-I$ and $g(JX,JY)=g(X,Y)$, $(X,Y\in\mathfrak{X}(M))$. Such a manifold (M^{2n},g,J) is called to be an almost complex manifold. A tensor field N of type (1,2) on almost complex manifold (M^{2n},g,J) given by

$$N(X,Y) := J([X,Y])-[JX,Y]-[X,JY]-J([JX,JY]), \tag{2.5}$$

$X, Y \in \mathfrak{X}(M^{2n})$, is called to be Nijenhuis tensor field. Almost complex manifold (M^{2n}, g, J) becomes an n-dimensional complex manifold if and only if

$$(2.6) \quad N(X,Y) = 0, \quad (X, Y \in \mathfrak{X}(M)).$$

3. Proofs of Main Theorem.

We preserve the notations appeared in Theorem 1. We put $G := R^{\times} SU(2)$, $\mathfrak{m} := T_e(R^{\times} SU(2))$ and $c := \sqrt{2} a_1 a_2 a_3$. We get from (1.1)

$$(3.1) \quad \begin{cases} [X_1,\ X_2] = {a_3}^2 c^{-1} X_3, & [X_2,\ X_3] = {a_1}^2 c^{-1} X_1, \\ [X_3,\ X_1] = {a_2}^2 c^{-1} X_2, & [X_i,\ X_4] = 0 \quad (i = 1,2,3,4). \end{cases}$$

From the construction of $g(a)_e$ in $(G, g(a))$ of Theorem 1, (2.1) and (3.1), $\mathfrak{m}$-valued symmetric function U on $\mathfrak{m} \times \mathfrak{m}$ is given as follows:

$$(3.2) \quad \begin{cases} U(X_i,\ X_i) = U(X_i,\ X_4) \ = \ 0 \quad (i = 1,2,3,4), \\ U(X_1,\ X_2) = (-{a_1}^2 + {a_2}^2)(2c)^{-1} X_3, \\ U(X_2,\ X_3) = (-{a_2}^2 + {a_3}^2)(2c)^{-1} X_1, \\ U(X_3,\ X_1) = (-{a_3}^2 + {a_1}^2)(2c)^{-1} X_2. \end{cases}$$

From (2.4) and (3.2), the proof of Theorem 1 is completed.

We get from (2.1), (3.1) and (3.2)

$$(3.3) \quad \begin{cases} \alpha(X_i, X_i) = \alpha(X_i, X_4) = \alpha(X_4, X_i) = 0 \quad (i = 1,2,3,4), \\ \alpha(X_1, X_2) = (-{a_1}^2 + {a_2}^2 + {a_3}^2)(2c)^{-1} X_3, \\ \alpha(X_1, X_3) = ({a_1}^2 - {a_2}^2 - {a_3}^2)(2c)^{-1} X_2, \\ \alpha(X_2, X_1) = (-{a_1}^2 + {a_2}^2 - {a_3}^2)(2c)^{-1} X_3, \\ \alpha(X_2, X_3) = ({a_1}^2 - {a_2}^2 + {a_3}^2)(2c)^{-1} X_1, \\ \alpha(X_3, X_1) = ({a_1}^2 + {a_2}^2 - {a_3}^2)(2c)^{-1} X_2, \\ \alpha(X_3, X_2) = (-{a_1}^2 - {a_2}^2 + {a_3}^2)(2c)^{-1} X_1. \end{cases}$$

Let R (resp. $S(g(a))$) be the Ricci tensor (resp. the scalar curvature) of $(G, g(a))$. Then we have from (2.3) and (3.3)

$$(3.4)\quad \begin{cases} R(X_1, X_1) = ({a_1}^4 - {a_2}^4 + 2{a_2}^2{a_3}^2 - {a_3}^4)/2c^2, \\ R(X_2, X_2) = (-{a_1}^4 + {a_2}^4 + 2{a_1}^2{a_3}^2 - {a_3}^4)/2c^2, \\ R(X_3, X_3) = (-{a_1}^4 - {a_2}^4 + 2{a_1}^2{a_2}^2 + {a_3}^4)/2c^2, \\ R(X_4, X_4) = 0. \end{cases}$$

We get from (3.4)

$$(3.5)\quad S(g(a)) = (-{a_1}^4 + 2{a_1}^2{a_2}^2 - {a_2}^4 + 2{a_2}^2{a_3}^2 - {a_3}^4 + 2{a_3}^2{a_1}^2)/2c^2.$$

By Thoerem 1 and (3.5), the proof of Corollary 2 is completed.

Using (2.5) and (3.1), we have

$$(3.6)\quad \begin{cases} N(X_1, X_2) = ({a_2}^2 - {a_3}^2)c^{-1}X_2, & N(X_1, X_3) = (-{a_2}^2 + {a_3}^2)c^{-1}X_3, \\ N(X_1, X_4) = N(X_2, X_3) = 0, & N(X_2, X_4) = ({a_2}^2 - {a_3}^2)c^{-1}X_3, \\ N(X_3, X_4) = ({a_2}^2 - {a_3}^2)c^{-1}X_2. & \end{cases}$$

By (2.6) and (3.6), the proof of Theorem 3 is completed. Moreover, from Theorem 1 and Theorem 3 we can get Corollary 4.

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