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A COMPLETENESS ON GENERALIZED FIBONACCI SEQUENCES

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1. Introduction

Let $V = (v_1, v_2, \cdots)$ be a sequence of positive integers arranged in nondecreasing order. We define V to be *complete* if every positive integer n is the sum of some subsequence of V, that is,

$$n = \sum_{i=1}^{\infty} a_i v_i \text{ where } a_i = 0 \text{ or } 1. \tag{1.1}$$

For example, let the Fibonacci sequence be denoted by F:

$$F = (F_1, F_2, F_3, \cdots) = (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots),$$

where $F_0 = 0$, $F_1 = F_2 = 1$, and, for $n > 2$, $F_n = F_{n-1} + F_{n-2}$.

John L. Brown, Jr. gave a criterion for completeness in 1961:

BROWN'S CRITERION. *A sequence V is complete if and only if*

(i) $v_1 = 1$ *and*
(ii) *for all* $n = 1, 2, \cdots$
$s_{n-1} = v_1 + v_2 + \cdots + v_{n-1} \geq v_n - 1$.

COROLLARY A. *If $v_1 = 1$ and $v_{n+1} \leq 2v_n$, then V is complete.*

From the above corollary, we can easily show that, the well known fact, the Fibonacci sequence is complete.

A sequence V, although unable to produce some numbers at the begining, might be able to generate all numbers beyond some point N.

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Such sequence, we shall say that they are *weakly complete* in contrast to the *strongly complete* sequences which are capable of generating all positive integers.

Now, we define the *k-generalized Fibonacci sequence* $\{g_n^{(k)}\}$ as follows:

$$g_1^{(k)} = g_2^{(k)} = \cdots = g_{k-2}^{(k)} = 0,\ g_{k-1}^{(k)} = g_k^{(k)} = 1, \text{ and for } n > k \geq 2$$

$$g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots g_{n-k}^{(k)}. \tag{1.2}$$

The above number $g_n^{(k)}$ is called the *nth k-generalized Fibonacci number*. Let $G^{(k)} = (1, 1, 2, 4, 8, 16, 32, \cdots)$ for k-generalized Fibonacci sequence $\{g_n^{(k)}\}$. For example, if $k = 7$, then $g_1^{(7)} = \cdots = g_5^{(7)} = 0$, $g_6^{(7)} = g_7^{(7)} = 1$, and then the sequence of 7-generalized Fibonacci numbers is

$$G^{(7)}$$
$$=(1, 1, 2, 4, 8, 16, 32, 64, 127, 253, 504, 1004, 2000, 3984, 7936, 15808, \cdots).$$

The *permanent* of an n-square matrix $A = [a_{ij}]$ is defined by

$$\text{per } A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}, \tag{1.3}$$

where the summation extends over all permutations σ of the symmetric group S_n. A matrix is said to be a (0,1)-matrix if each of its entries is either 0 or 1.

Let $\mathcal{F}^{(n,k)} = [f_{ij}]$ be the $n \times n$ (0,1)-$(k+1)$st *super diagonal matrix* defined by

$$\mathcal{F}^{(n,k)} = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 \\ & & & & & & & & \ddots & \vdots \\ & & \ddots & \ddots & & & & & \ddots & 0 \\ \vdots & & & \ddots & \ddots & & & \ddots & & 1 \\ \vdots & & & & \ddots & \ddots & & & & \vdots \\ 0 & & & \cdots & \cdots & & & 0 & 1 & 1 \end{bmatrix}, \tag{1.4}$$

where $f_{11} = \cdots = f_{1k} = 1$ and $f_{1k+1} = \cdots = f_{1n} = 0$.

In [3], the author determined the relationship between $g_n^{(k)}$ and the permanent of the $n \times n$ (0,1)-$(k+1)$st super diagonal matrix $\mathcal{F}^{(n,k)}$ by using the matrix contraction.

THEOREM 1.1. [3] *Let $g_{n+1}^{(k)}$ be the $(n+1)$st k-generalized Fibonacci number, $n \geq k$. Then*

$$\text{per}\, \mathcal{F}^{(n,k)} = g_{n+k-1}^{(k)}. \tag{1.5}$$

And, in [3], the next theorem is a matrix which is not a tridiagonal matrix whose permanent equals to the $(n+1)$st Fibonacci number.

THEOREM 1.2. [3] *Let*

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & 1 & \cdots & 1 \\ & & & & & \\ \vdots & & & \ddots & & \vdots \\ & & & & 0 & 1 \\ 0 & & \cdots & 0 & 1 & 1 \end{bmatrix}_{n \times n} \tag{1.6}$$

Then

$$\text{per}\, P^T U P = F_{n+1} \tag{1.7}$$

for any permutation matrix P.

2. Completeness on $G^{(k)}$

Now, we consider the completeness on $G^{(k)}$. Let

$$G^{(k)} = (1, 1, 2, 4, 8, \cdots) = (g_1^{(k)}, g_2^{(k)}, g_3^{(k)}, \cdots)$$

be the k-generalized Fibonacci sequence, $k \geq 2$. The following theorem shows the completeness on k-generalized Fibonacci sequences.

THEOREM 2.1. *The sequence $G^{(k)}$ is complete, $k \geq 2$.*

Proof. Clearly $g_1^{(k)} = 1$. We proceed by induction on n, if $n = 2$, then $g_2^{(k)} = 1 \leq 2g_1^{(k)}$. Assume true for $n-1$, $g_{n-1}^{(k)} \leq 2g_{n-2}^{(k)}$, and consider for n.

$$\begin{aligned}
&(g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}) + g_{n-k-1}^{(k)} \\
&\quad \leq g_{n-1}^{(k)} + g_{n-2}^{(k)} + (g_{n-3}^{(k)} + \cdots + g_{n-k-2}^{(k)}) \\
&\Longleftrightarrow g_n^{(k)} + g_{n-k-1}^{(k)} \leq g_{n-1}^{(k)} + (g_{n-2}^{(k)} + \cdots + g_{n-k-1}^{(k)}) + g_{n-k-2}^{(k)} \\
&\quad = g_{n-1}^{(k)} + g_{n-1}^{(k)} + g_{n-k-2}^{(k)} \\
&\quad = 2g_{n-1}^{(k)} + g_{n-k-2}^{(k)}.
\end{aligned}$$

Since $g_{n-k-1}^{(k)} \geq g_{n-k-2}^{(k)}$, $g_n^{(k)} \leq 2g_{n-1}^{(k)}$. Therefore, by Corollary A, $G^{(k)}$ is complete.

In the Fibonacci sequence F, $F - F_r$ is complete. What can we say about the completeness on $G^{(k)} - g_r^{(k)}$? The following theorem is an answer for that.

THEOREM 2.2. *Let $k \geq 3$. $G^{(k)} - g_r^{(k)}$ is complete for $0 < r \leq 2$, and $G^{(k)} - g_r^{(k)}$ is not complete for $r \geq 3$.*

Proof. Suppose that $0 < r \leq 2$. Since $g_1^{(k)} = g_2^{(k)}$, without loss of generality, let $r = 1$. Then $G^{(k)} - g_r^{(k)} = (g_2^{(k)}, g_3^{(k)}, \cdots)$ and the first term in $G^{(k)} - g_r^{(k)}$ is still 1. If $n \leq k$, then $g_2^{(k)} + g_3^{(k)} + \cdots + g_n^{(k)} = g_{n+1}^{(k)}$. So, $g_2^{(k)} + g_3^{(k)} + \cdots + g_n^{(k)} \geq g_{n+1}^{(k)} - 1$. Thus $G^{(k)} - g_r^{(k)}$ is complete. Now suppose that $n > k$. By induction on n,

$$g_2^{(k)} + g_3^{(k)} + \cdots + g_{k+1}^{(k)} = g_{k+2}^{(k)} \geq g_{k+2}^{(k)} - 1.$$

Assume true for n, i.e., $g_2^{(k)} + \cdots + g_n^{(k)} \geq g_{n+1}^{(k)} - 1$. Then

$$\begin{aligned}
g_2^{(k)} + \cdots + g_n^{(k)} + g_{n+1}^{(k)} &\geq 2g_{n+1}^{(k)} - 1 \\
&= g_{n+1}^{(k)} + (g_n^{(k)} + g_{n-1}^{(k)} + \cdots + g_{n-k+1}^{(k)}) - 1 \\
&= g_{n+2}^{(k)} + g_{n-k+1}^{(k)} - 1 \\
&\geq g_{n+2}^{(k)} - 1.
\end{aligned}$$

Thus, if $0 < r \leq 2$, then $G^{(k)} - g_r^{(k)}$, $k \geq 3$, is complete.

Now we suppose that $r \geq 3$. First, we prove that $g_1^{(k)} + g_2^{(k)} + \cdots + g_{r-1}^{(k)} < g_{r+1}^{(k)} - 1 < g_{r+1}^{(k)}$ for all $r \geq 3$. Assume that $3 \leq r \leq k$. Then, clearly,

$$g_1^{(k)} + g_2^{(k)} + \cdots + g_{r-1}^{(k)} = g_r^{(k)} < g_{r+1}^{(k)} - 1 < g_{r+1}^{(k)}.$$

Now, assume that $r > k$. By induction on r, clearly,

$$g_1^{(k)} + g_2^{(k)} + \cdots + g_k^{(k)} = g_{k+1}^{(k)} < g_{k+2}^{(k)} - 1 < g_{k+2}^{(k)}.$$

Assume true for r and consider $r+1$. That is,

$$\begin{aligned} g_1^{(k)} + \cdots + g_{r-1}^{(k)} + g_r^{(k)} &< g_r^{(k)} + g_{r+1}^{(k)} - 1 \\ &= g_{r-1}^{(k)} + g_{r-2}^{(k)} + \cdots + g_{r-k}^{(k)} + g_{r+1}^{(k)} - 1 \\ &= g_{r+1}^{(k)} + g_r^{(k)} + g_{r-1}^{(k)} + \cdots + g_{r-k}^{(k)} - 1 - g_r^{(k)} \\ &= g_{r+2}^{(k)} - 1 - g_r^{(k)} + g_{r-k}^{(k)} \\ &< g_{r+2}^{(k)} - 1 < g_{r+2}^{(k)}. \end{aligned}$$

Thus, in any cases, $g_1^{(k)} + \cdots + g_{r-1}^{(k)} < g_{r+1}^{(k)} - 1 < g_{r+1}^{(k)}$ for all $r \geq 3$. This result is same that $g_{r+1}^{(k)} - 1$ is unattainable as a sum of terms in a subsequence of $G^{(k)} - g_r^{(k)}$, $r \geq 3$. Therefore, $G^{(k)} - g_r^{(k)}$, $r \geq 3$, is not complete.

In the Fibonacci sequence F, $F - F_r - F_s$ is not even weakly complete. In fact, $G^{(k)} - g_r^{(k)} - g_s^{(k)}$, $s < r$, is not complete. For example, if $r = 2$ and $s = 1$, then $G^{(k)} - g_2^{(k)} - g_1^{(k)}$ does not contains 1. Thus $G^{(k)} - g_r^{(k)} - g_s^{(k)}$ can never be complete. Then will $G^{(k)} - g_r^{(k)} - g_s^{(k)}$ be good enough to be weakly complete? An answer can be given as following:

Theorem 2.3. *The sequence $G^{(k)} - g_{k+1}^{(k)} - g_{2k+1}^{(k)}$ is not weakly complete, $k \geq 2$.*

Proof. If $k = 2$, then the proof is completed. Now, suppose that $k \geq 3$. We easily see that the number $g_{2k+1}^{(k)} + 1$ is unattainable as a sum of terms in a subsequence of $G^{(k)} - g_{k+1}^{(k)} - g_{2k+1}^{(k)}$. That is,

$$g_1^{(k)} + \cdots + g_k^{(k)} + g_{k+2}^{(k)} + \cdots + g_{2k}^{(k)} = g_{k+1}^{(k)} + g_{k+2}^{(k)} + \cdots + g_{2k}^{(k)} < g_{2k+1}^{(k)} + 1 < g_{2k+2}^{(k)}.$$

We will use this result as the basis of an induction that the number $n = g_{2k+1}^{(k)} + g_{kt}^{(k)} + 1$ is unattainable for all $t = 1, 2, 3, \cdots$. The result is established, then, for $t = 1$. That is, the number $g_{2k+1}^{(k)} + g_k^{(k)} + 1$ is unattainable as a sum of a subsequence of $G^{(k)} - g_{k+1}^{(k)} - g_{2k+1}^{(k)}$. Suppose, for some value $t \geq 1$, that the number $n = g_{2k+1}^{(k)} + g_{kt}^{(k)} + 1$ is unattainable as a sum of a subsequence of $G^{(k)} - g_{k+1}^{(k)} - g_{2k+1}^{(k)}$. Consider now the number $g_{2k+1}^{(k)} + g_{k(t+1)}^{(k)} + 1$, that is, $n = g_{2k+1}^{(k)} + g_{kt+k}^{(k)} + 1$. After the disposal of $g_{k+1}^{(k)}$ and $g_{2k+1}^{(k)}$, the numbers we have display

$$g_1^{(k)}, g_2^{(k)}, \cdots, g_k^{(k)}, g_{k+2}^{(k)}, \cdots, g_{2k}^{(k)}, g_{2k+2}^{(k)}, \cdots, g_{tk-1}^{(k)}, g_{tk}^{(k)}, \cdots, g_{tk+k}^{(k)}.$$

The number $g_{2k+1}^{(k)} + g_{tk+k}^{(k)}$ is attainable and

$$g_{2k+1}^{(k)} + g_{tk+k}^{(k)} < g_{2k+1}^{(k)} + g_{tk+k}^{(k)} + 1 < g_{tk+k+1}^{(k)}.$$

With the number $g_{2k+1}^{(k)} + g_{tk+k}^{(k)} + 1$ in hand, we get

$$g_{2k+1}^{(k)} + g_{tk+k}^{(k)} + 1 - (g_{tk+1}^{(k)} + \cdots g_{tk+(k-1)}^{(k)}) = g_{2k+1}^{(k)} + g_{tk}^{(k)} + 1,$$

is unattainable where the $g_{tk+1}^{(k)} + \cdots + g_{tk+(k-1)}^{(k)}$ is attainable. Therefore, if $g_{2k+1}^{(k)} + g_{tk}^{(k)} + 1$ is unattainable, so is $g_{2k+1}^{(k)} + g_{tk+k}^{(k)} + 1$. By induction, then, $g_{2k+1}^{(k)} + g_{tk}^{(k)} + 1$ is unattainable for all $t = 1, 2, 3, \cdots$. Since there are numbers $g_{2k+1}^{(k)} + g_{tk}^{(k)} + 1$ which exceed every choice of positive integer, the sequence $G^{(k)} - g_{k+1}^{(k)} - g_{2k+1}^{(k)}$ is not even weakly complete.

By the above theorem, $G^{(k)} - g_r^{(k)} - g_s^{(k)}$ is not even weakly complete, in general. We consider some k-generalized Fibonacci sequence $G^{(k)}$;

i) $G^{(4)} = (1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, 2872, 5536, 10671, \cdots)$

ii) $G^{(5)} = (1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, 3525, 6930, 13624, \cdots)$

iii) $G^{(6)} = (1, 1, 2, 4, 8, 16, 32, 63, 125, 248, 492, 976, 1936, 3840, 7617, 15109, \cdots)$

iv) $G^{(7)} = (1, 1, 2, 4, 8, 16, 32, 64, 127, 253, 504, 1004, 2000, 3984, 7936, 15808, \cdots)$

v) $G^{(8)} = (1, 1, 2, 4, 8, 16, 32, 64, 128, 255, 509, 1016, 2028, 4048, 8080, 16128, \cdots)$

For example, consider

$$G^{(4)} - 8 - 108$$
$$=(1, 1, 2, 4, 15, 29, 56, 208, 401, 773, 1490, 2872, 5536, 10671, \cdots).$$

882 is unattainable as a sum of terms in a subsequence of $G^{(4)} - 8 - 108$, since $882 = g_9^{(4)} + g_{12}^{(4)} + 1$.

Let L_n be the nth Lucas number. That is, $L_n = F_{n+1} + F_{n-1}$ for all $n = 1, 2, 3, \cdots$. Thus we have $L_1 = 1, L_2 = 3, L_3 = 4$, and so on. Since the Fibonacci numbers are connected by the fundamental recursion $F_n = F_{n-1} + F_{n-2}$, it follows immediately that the Lucas numbers are likewise related:

$$L_n = L_{n-1} + L_{n-2} \text{ for } n > 2. \tag{2.1}$$

Let $G_\dagger^{(k)} = (g_0^{(k)}, g_1^{(k)}, g_2^{(k)}, \cdots)$ be the k-generalized Fibonacci sequence such that $g_0^{(k)} = g_1^{(k)} = \cdots = g_{k-2}^{(k)} = 0, g_{k-1}^{(k)} = g_k^{(k)} = 1$. Now, we define the *$k$-generalized Lucas sequence* $\{l_n^{(k)}\}$ by the following as

$$l_{n+1}^{(k)} = g_n^{(k)} + g_{n+k}^{(k)}, \ n = 0, 1, 2, \cdots. \tag{2.2}$$

That is, $l_1^{(k)} = g_0^{(k)} + g_k^{(k)}$, $l_2^{(k)} = g_1^{(k)} + g_{k+1}^{(k)} \cdots, l_k^{(k)} = g_{k-1}^{(k)} + g_{2k-1}^{(k)}$.

Then, for $n > k$

$$\begin{aligned} l_{n+1}^{(k)} &= g_n^{(k)} + g_{n+k}^{(k)} = (g_{n-1}^{(k)} + \cdots + g_{n-k}^{(k)}) + (g_{n+k-1}^{(k)} + \cdots + g_{n+k-k}^{(k)}) \\ &= (g_{n-1}^{(k)} + g_{n+k-1}^{(k)}) + \cdots + (g_{n-k}^{(k)} + g_n^{(k)}) \\ \text{(2.3)} \quad &= l_n^{(k)} + l_{n-1}^{(k)} + \cdots l_{n-k+1}^{(k)}. \end{aligned}$$

THEOREM 2.4. *The k-generalized Lucas sequence is not weakly complete for $k \geq 2$.*

Proof. Let $\mathcal{L}^{(k)} = (l_1^{(k)}, l_2^{(k)}, l_3^{(k)}, \cdots)$ be the k-generalized Lucas sequence. Since $l_1^{(k)} + \cdots + l_{k-1}^{(k)} = g_{2k-1}^{(k)} - g_{k-1}^{(k)} < g_{2k-1}^{(k)} < l_k^{(k)}$, $g_{2k-1}^{(k)}$ is unattainable as a sum of a subsequence of $\mathcal{L}^{(k)}$. We will use this result as the basis of an induction that the number $m = l_n^{(k)} + g_{2k-1}^{(k)}$ is unattainable for $n \geq k+1$. By induction on n, if $n = k+1$, $l_1^{(k)} + \cdots + l_k^{(k)} = l_{k+1}^{(k)} < l_{k+1}^{(k)} + g_{2k-1}^{(k)}$. Since

$$l_{k+2}^{(k)} = l_{k+1}^{(k)} + l_k^{(k)} + \cdots + l_2^{(k)} = l_{k+1}^{(k)} + (g_{k-1}^{(k)} + g_{2k-1}^{(k)}) + l_{k-1}^{(k)} + \cdots + l_2^{(k)},$$

$l_{k+2}^{(k)} > l_{k+1}^{(k)} + g_{2k-1}^{(k)}$. So, $l_1^{(k)} + \cdots + l_k^{(k)} = l_{k+1}^{(k)} < l_{k+1}^{(k)} + g_{2k-1}^{(k)} < l_{k+2}^{(k)}$. Thus, $l_{k+1}^{(k)} + g_{2k-1}^{(k)}$ is unattainable as a sum of a subsequence of $\mathcal{L}^{(k)}$. Suppose, for some value $n > k+1$, that the number $m = l_{n+1}^{(k)} + g_{2k-1}^{(k)}$. The number $l_{n+1}^{(k)}$ is attainable and

$$l_{n-(k-1)}^{(k)} + \cdots + l_n^{(k)} = l_{n+1}^{(k)} < l_{n+1}^{(k)} + g_{2k-1}^{(k)} < l_{n+2}^{(k)}.$$

With the number $l_n^{(k)} + g_{2k-1}^{(k)}$ in hand, we get

$$l_n^{(k)} + g_{2k-1}^{(k)} = (l_{n+1}^{(k)} + g_{2k-1}^{(k)}) - (l_{n-1}^{(k)} + \cdots + l_{n-(k-1)}^{(k)}),$$

is unattainable where the $l_{n-1}^{(k)} + \cdots + l_{n-(k-1)}^{(k)}$ is attainable. Therefore, if $l_n^{(k)} + g_{2k-1}^{(k)}$ is unattainable, so is $l_{n+1}^{(k)} + g_{2k-1}^{(k)}$. By induction, then, $l_n^{(k)} + g_{2k-1}^{(k)}$ is unattainable for all $n \geq k+1$. The proof is completed.

3. Other Results

In case $k = 3$, the fundamental recurrence relation $g_{n+1}^{(3)} = g_n^{(3)} + g_{n+1}^{(3)} + g_{n-2}^{(3)}$ can also be defined as the vector recurrence relation

$$(3.1) \qquad \begin{pmatrix} g_{n-1}^{(3)} \\ g_n^{(3)} \\ g_{n+1}^{(3)} \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} g_{n-2}^{(3)} \\ g_{n-1}^{(3)} \\ g_n^{(3)} \end{pmatrix}$$

which is visibly equivalent. In terms of the 3×3 matrix

$$(3.2) \qquad Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

applying (3.1) n times, we have

$$(3.3) \qquad \begin{pmatrix} g_{n+1}^{(3)} \\ g_{n+2}^{(3)} \\ g_{n+3}^{(3)} \end{pmatrix} = Q^n \begin{pmatrix} g_1^{(3)} \\ g_2^{(3)} \\ g_3^{(3)} \end{pmatrix}.$$

Similarly, for the k-generalized Fibonacci sequence, the matrix and the vector recurrence relation is;

$$(3.4) \qquad \begin{pmatrix} g_{n+1}^{(k)} \\ g_{n+2}^{(k)} \\ \vdots \\ g_{n+k}^{(k)} \end{pmatrix} = Q^n \begin{pmatrix} g_1^{(k)} \\ g_2^{(k)} \\ \vdots \\ g_k^{(k)} \end{pmatrix},$$

where

$$(3.5) \qquad Q = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}_{k \times k}.$$

We have the following theorem by using the above facts.

THEOREM 3.1. *For any positive integers $n \geq k, m \geq k$,*

$$g_{n+m}^{(k)} = g_n^{(k)} g_{m-(k-1)}^{(k)} + (g_n^{(k)} + g_{n-1}^{(k)}) g_{m-(k-2)}^{(k)} g_{n-2}^{(k)}) g_{m-(k-3)}^{(k)} + (g_n^{(k)} + g_{n-1}^{(k)} + \cdots + g_{n+1}^{(k)} g_m^{(k)},$$

Proof. For $G^{(k)}$, $k \geq 2$, since $g_1^{(k)} = 1$ and $g_2^{(k)} = 1$, we can replace the matrix Q in (3.5) with

$$(3.6) \qquad Q = \begin{bmatrix} 0 & g_1^{(k)} & 0 & \cdots & 0 \\ 0 & 0 & g_1^{(k)} & \cdots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & g_1^{(k)} \\ g_1^{(k)} & g_1^{(k)} & \cdots & g_1^{(k)} & g_2^{(k)} \end{bmatrix}_{k \times k} .$$

Then

$Q^n =$

$$\begin{bmatrix} g_{n-(k-1)}^{(k)} & g_{n-(k-1)}^{(k)} + g_{n-k}^{(k)} & g_{n-(k-1)}^{(k)} + g_{n-k}^{(k)} + g_{n-k-1}^{(k)} & \cdots & g_{n-(k-2)}^{(k)} \\ g_{n-(k-2)}^{(k)} & g_{n-(k-2)}^{(k)} + g_{n-(k-1)}^{(k)} & g_{n-(k-2)}^{(k)} + g_{n-(k-1)}^{(k)} + g_{n-k}^{(k)} & \cdots & g_{n-(k-3)}^{(k)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{n-1}^{(k)} & g_{n-1}^{(k)} + g_{n-2}^{(k)} & g_{n-1}^{(k)} + g_{n-2}^{(k)} + g_{n-3}^{(k)} & \cdots & g_n^{(k)} \\ g_n^{(k)} & g_n^{(k)} + g_{n-1}^{(k)} & g_n^{(k)} + g_{n-1}^{(k)} + g_{n-2}^{(k)} & \cdots & g_{n+1}^{(k)} \end{bmatrix}$$

Since $Q^n Q^m = Q^{n+m}$, $g_{n+m}^{(k)} = (Q^{n+m})_{k1}$.

In the Fibonacci numbers, $F_n \mid F_{tn}$ for all $t = 1, 2, 3, \cdots$, since

$$(3.7) \qquad F_{n+m} = F_{n-1} F_m + F_n F_{m+1}.$$

What can we think about the divisibility on k-generalized Fibonacci numbers? For example, consider $g_6^{(7)}$ and $g_{12}^{(7)}$ in the 7-generalized Fibonacci sequence $\{g_n^{(7)}\}$. $6|12$ but $16 \not| 1004$. And, consider $g_4^{(6)}, g_8^{(6)}$ and $g_{16}^{(6)}$ in the 6-generalized Fibonacci sequence. $4|8$ and $4|16$ but $4 \not| 63$ and $4 \not| 15109$. Thus, we have established the following theorem.

THEOREM 3.2. *For the k-generalized Fibonacci sequence, there exists positive integer t such that* $g_n^{(k)} \not| g_{tn}^{(k)}$.

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