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L^2–FIELDS PRESERVING THE RICCI FIELD

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1. Introduction

The following result is well-known ([1]) : Let (M, g) be a connected, orientable and compact Riemannian manifold without boundary, and let Ric be the Ricci tensor field on M. If a conformal (or, projective) vector field Y on M satisfies $\Theta(Y)Ric = 0$, then Y is a Killing vector field on M, where $\Theta(Y)$ denotes the Lie differentiation with respect to Y. We consider the case of that M is a complete non-compact Riemannian manifold, and we have the following theorems :

THEOREM A. *Let (M, g) be an $n(\geq 2)$-dimensional complete non-compact Riemannian manifold. If an L^2-conformal field Y with L^2-characteristic function satisfies $\Theta(Y)Ric = 0$, then Y is a Killing field.*

THEOREM B. *Let (M, g) be as Theorem A. If an L^2- projective field Y with an L^2-characteristic form ϕ satisfies $\Theta(Y)Ric = 0$, then Y is a Killing field.*

We shall be in C^∞-category. The manifolds considered in this note are connected and orientable.

2. Preliminaries

Let M be a complete non-compact Riemannian manifold of dimension n. The metric tensor, the Riemannian connection and the Ricci field of M are respectively denoted by g, ∇ and Ric.

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DEFINITION. ([2],[3],[4],[6],[7]) If Y satisfies $\Theta(Y)g = 2fg$, where f is a function on M, then Y is called a conformal field and f the characteristic function of Y. If Y satisfies $\Theta(Y)g = 0$, then Y is called a Killing field. If Y satisfies

$$(\Theta(Y)\nabla)_X Z = \phi(X)Z + \phi(Z)X$$

for any $X, Z \in \Gamma(TM)$, where ϕ is a 1-form on M and $\Gamma(TM)$ denotes the sections of the tangent bundle TM of M, then Y is called a projective field and ϕ the characteristic form of Y. If Y satisfies $\Theta(Y)\nabla = 0$, then Y is called a affine field.

Let $\Lambda^s(M)(s = 0, 1, 2, \cdots, n)$ be the space of all s-forms on M and $\Lambda_0^s(M)$ be the subspace of $\Lambda^s(M)$ composed of forms with compact supports. Let $<< \ , \ >>$ be the global scalar product on $\Lambda^s(M)$. We denote by $L_s^2(M)$ the completion of $\Lambda_0^s(M)$ with respect to $<< \ , \ >>$ (cf. [5],[6],[7]). We notice that $\Lambda^0(M)$ denotes the space of all functions on M.

DEFINITION. (cf. [5],[6],[7]) An element $\eta \in L_s^2(M) \cap \Lambda^s(M)$ is called an L^2- field of M.

We notice that a function f on M has finite L^2-norm if $f \in L_0^2(M) \cap \Lambda^0(M)$. The Laplacian Δ acting on $\Lambda^s(M)$ is defined by

$$\Delta = \delta d + d\delta$$

(cf. [3]). We notice that the operator δ is the adjoint operator of the exterior derivative d with respect to the global scalar product $<< \ , \ >>$.

LEMMA 1. ([6]) *Let M be a complete non-compact Riemannian manifold. Every L^2-affine field on M is a Killing field.*

Let 0 be a point of M and fix it. We denote by $\rho(x)$ the geodesic distance from 0 to $x \in M$. We set

$$B(k) = \{x \in M | \rho(x) < k\}$$

for any $k > 0$. There exists family $\{\omega_k\}_{k>0}$ of Lipschitz continuous functions on M satisfying the follows:

$$\begin{cases} 0 \leq \omega_k(x) \leq 1 & \text{for } x \in M, \\ supp\omega_k \subset B(2k), & \\ \omega_k(x) = 1 & \text{for } x \in B(k), \\ \lim_{k\to\infty} \omega_k = 1, & \\ |d\omega_k| \leq Ck^{-1} & \text{almost everywhere on } M, \end{cases}$$

where $C > 0$ is a constant independent of k (cf. [1],[5],[6],[7]).

LEMMA 2. (cf. [1],[5],[6],[7]) *For any $\eta \in \Lambda^s(M)$, there exists a positive constant A independent of k such that*

$$||d\omega_k \otimes \eta||^2_{B(2k)} \leq \frac{A}{k^2}||\eta||^2_{B(2k)},$$

where

$$||\eta||^2_{B(2k)} = << \eta, \eta >>_{B(2k)} = \int_{B(2k)} < \eta, \eta > dS$$

and dS denotes the volume element of $B(2k)$.

If η has finite L^2-norm, then $\omega_k\eta \in \Lambda^*_0(M)$ and $\omega_k\eta \to \eta (k \to +\infty)$ in the strong sense.

We finally prepare the following theorem for later use.

LEMMA 3. ([1]) *Let (M, g) be a complete non-compact Riemannian manifold. Then*

$$\int_{B(2k)} div(\omega_k Y) dS = 0$$

for any $Y \in \Gamma(TM)$, where dS denotes the volume element of $B(2k)$.

3. Proof of main theorems

Using the lemmas in the previous section, we will prove the theorems mentioned in the introduction

THEOREM A. *Let (M, g) be an $n(\geq 2)$-dimensional complete non-compact Riemannian manifold. If an L^2-conformal field Y with L^2-characteristic function satisfies $\Theta(Y)Ric = 0$, then Y is a Killing field.*

Proof. Let Y be an L^2-conformal field of M with L^2-characteristic function f. Then it follows that

$$g^{ji}(\Theta(Y)Ric)(X_j, X_i) = 2(n-1)\Delta f,$$

where $\{X_i\}$ denotes an orthonormal fields and $g_{ji} = g(X_j, X_i), (g_{ji})^{-1} = (g^{ji})$ ([cf. [2],[3],[4]). So we obtain directly $\Delta f = 0$ by the assumption $\Theta(Y)Ric = 0$ and $n \geq 2$.

By means of Lemma 2 we can see that f is constant on M. In fact, we have

$$\begin{aligned} 0 &= << \Delta f, \omega_k^2 f >>_{B(2k)} \\ &= << \omega_k df, \omega_k df >>_{B(2k)} + 2 << \omega_k df, f d\omega_k >>_{B(2k)} \\ &\geq \|\omega_k df\|^2_{B(2k)} - 2\|\omega_k df\|_{B(2k)} \|f d\omega_k\|_{B(2k)} \\ &\geq \frac{3}{4}\|\omega_k df\|^2_{B(2k)} - \frac{4C^2}{k^2}\|f\|^2_{B(2k)}. \end{aligned}$$

Since f is an L^2-function on M, we have $df = 0$ as $k \to \infty$. Therefore f is constant on M. Moreover, for any conformal field Y we have

$$div(\omega_k Y) = q\omega_k f + g(Y, grad\omega_k)$$

because of $divY = nf$. Thus we obtain

$$\begin{aligned} &\int_{B(2k)} div(\omega_k Y) dS \\ &\geq nf \int_{B(2k)} \omega_k dS - \int_{B(2k)} |Y||grad\omega_k| dS \\ &\geq nf \int_{B(2k)} \omega_k dS - \frac{C}{k} \int_{B(2k)} |Y| dS. \end{aligned}$$

From Lemma 3, we get $f \leq 0$ as $k \to \infty$.

Since $0 = \int_{B(2k)} div(\omega_k Y)dS \leq nf \int_{B(2k)} \omega_k dS + \frac{C}{k}\int_{B(2k)} |Y| dS$, we obtain $f \geq 0$ as $k \to \infty$. And hence f vanishes identically. Thus Y is a Killing field of M.

THEOREM B. *Let (M, g) be as Theorem A. If an L^2- projective field Y with an L^2-characteristic form ϕ satisfies $\Theta(Y)Ric = 0$, then Y is a Killing field.*

Proof. For any projective field Y with characteristic form ϕ, it follows that

$$g^{ji}(\Theta(Y)Ric)(X_j), X_i) = (n-1)\delta(\phi)$$

(cf. [2],[3],[4]). Since $n \geq 2$ and $\Theta(s)Ric_D = 0$, we have $\delta\phi = 0$, which together with $d(divY) = (n+1)\phi$ implies

$$0 = \delta\phi = \frac{1}{n+1}\delta d(divY) = \frac{1}{n+1}\Delta(divY).$$

Since ϕ is an L^2-form on M, $divY$ is also an L^2-function on M. If we recall the fact in the process of the proof of Theorem A, then $divY$ is constant on M and consequently $\phi = 0$, namely, Y is a affine field. By means of Lemma 1, we see that Y is a Killing field of M.

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