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A CHARACTERIZATION OF PROJECTIVE GEOMETRIES

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1. Introduction

The most fundamental examples of (combinatorial) geometries are projective geometries $PG(n-1,q)$ of dimension $n-1$, representable over $GF(q)$, where q is a prime power. Every upper interval of a projective geometry is a projective geometry. The Whitney numbers of the second kind are the Gaussian coefficients. Every flat of a projective geometry is modular, so the projective geometry is supersolvable in the sense of Stanley [6].

The characteristic polynomial $p(G,\lambda)$ of a geometry G of rank n is defined by

$$p(G,\lambda)=\sum_{a\in L(G)}\mu(\hat{0},a)\lambda^{n-r(a)}$$

where $L(G)$ is the lattice of flats of G and μ is the Möbius function of $L(G)$.

In this paper, we give a characterization of projective geometries in terms of their characteristic polynomials and some other conditions.

Our notation and terminology follow those in [7,8]. To clarify our terminology, let G be a finite geometric lattice. If S is the set of points (or rank-one flats) in G, the lattice structure of G induces the structure of a (combinatorial) geometry, also denoted by G, on S. The *size* $|G|$ of the geometry G is the number of points in G. Let T be a subset of S. The *deletion* of T from G is the geometry on the point set $S\backslash T$ obtained by restricting G to the subset $S\backslash T$. The *contraction* G/T of G by T is the geometry induced by the geometric lattice $[cl(T),\hat{1}]$ on

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the set S' of all flats in G covering $cl(T)$. (Here, $cl(T)$ is the closure of T, and $\hat{1}$ is the maximum of the lattice G.) Thus, by definition, the contraction of a geometry is always a geometry. A geometry which can be obtained from G by deletions or contractions is called a *minor* of G.

2. Preliminaries

A geometry G is said to be *upper homogeneous* if for $k = 1, 2, \ldots, r(G), G/x \cong G/y$ for every pairs x, y of flats of rank k. Kahn and Kung [4] defined splitting in geometries. A geometry G *splits* if G is the union of two of its proper flats. And G is said to be *non-splitting* otherwise.

LEMMA 2.1. [9] *If a geometry G is upper homogeneous, has a modular copoint, and $|G| > r(G)$, then G is non-splitting.*

LEMMA 2.2. *Let G be an upper homogeneous geometry having a modular copoint. Then G is supersolvable. Let $\emptyset < x_1 < x_2 < \cdots < x_{n-1} < x_n = G$ be a maximal chain of modular flats of G. Let a_i be the number of points in x_i but not in x_{i-1} for each $i = 2, 3, \ldots, n$. Then we have $a_i \leq a_{i+1}$ for each $i = 1, 2, \ldots, n-1$.*

Proof. Let n be the rank of G and let x_{n-1} be a modular copoint of G. Then $[\hat{0}, x_{n-1}] \cong G/a$ for a point a not in x_{n-1}. Since G is upper homogeneous, it follows that $[\hat{0}, x_{n-1}] \cong G/b$ for a point b in x_{n-1}. Thus x_{n-1} is upper homogeneous and has a modular copoint x_{n-2} of x_{n-1} such that $[\hat{0}, x_{n-2}] \cong x_{n-1}/b$. It follows that x_{n-2} is a modular coline of G. By repeating the same arguments, we have a maximal chain $\emptyset < x_1 < x_2 < \cdots < x_{n-1} < G$ of modular flats in G. Thus G is supersolvable. Let a be a point in x_i but not in x_{i-1} for some i. Since $x_{i+1}/a \cong [\hat{0}, x_i]$ and $x_i/a \cong [\hat{0}, x_{i-1}]$, it implies that $a_i = |x_i| - |x_{i-1}| \leq |x_{i+1}| - |x_{i+1}/a| = |x_{i+1}| - |x_i| = a_{i+1}$. Thus $a_i \leq a_{i+1}$ for each $i = 1, 2, \ldots, n-1$.

A geometry is *modular* if all of its flats are modular. The following propositions give characterizations of modular geometries.

PROPOSITION 2.3. [1] *A geometry is modular if and only if it is the direct sum of projective geometries or points.*

PROPOSITION 2.4. [3] *A geometry G is modular if and only if the number of points in G is the same as the number of copoints in G.*

The *Whitney numbers* of a geometry G of rank n are defined by

$$w(n,s) = \sum_{r(x)=n-s} \mu(\hat{0}, x),$$

the coefficient of λ^s in the characteristic polynomial ; and

$$W(n,s) = \sum_{r(x)=n-s} 1,$$

the number of flats of rank $n-s$. The most well-known examples are the following (See Dowling[2, p.75]) :

(1) If $G = B_n$, the Boolean algebra of rank n, then

$$w(n,s) = (-1)^{n-s}\binom{n}{s} \quad \textit{and} \quad W(n,s) = \binom{n}{s}.$$

(2) If $G = PG(n-1,q)$, then

$$w(n,s) = (-1)^{n-s} q^{\binom{n-s}{2}} \binom{n}{s}_q \quad \textit{and} \quad W(n,s) = \binom{n}{s}_q,$$

where $\binom{n}{s}_q$ is the *Gaussian coefficient,*

$$\binom{n}{s}_q = \frac{(q^n-1)\dots(q^{n-s+1}-1)}{(q^s-1)\dots(q-1)}.$$

Each of these examples are classes of geometries which satisfy the hypotheses of the following theorem due to Dowling.

THEOREM 2.5. [2] *Let $\{G_n : n = 1, 2, \dots\}$ be a class of geometries such that G_n is of rank n, and, for all flats x in G_n of rank $n-s$*

$(0 \le s \le n)$, the interval $[x, \hat{1}]$ is isomorphic to G_s. Let $w(n,s), W(n,s)$ be the Whitney numbers of G_n. Then

$$\sum_s W(n,s)w(s,t) = \delta(n,t),$$

$$\sum_s w(n,s)W(s,t) = \delta(n,t),$$

and the numbers $w(n,s), W(n,s)$ satisfy the inverse relations

$$a_n = \sum_s W(n,s)b_s, \quad b_n = \sum_s w(n,s)a_s.$$

3. Main Theorem

THEOREM 3.1. *Let q be a power of prime. If a geometry G is upper homogeneous, has a modular copoint, and $p(G;\lambda) = (\lambda - 1)\,(\lambda - q)(\lambda - q^2)\dots(\lambda - q^{n-1})$, then $G \cong PG(n-1, q)$.*

Proof. By Lemma 2.2, G is supersolvable. Let $\emptyset < x_1 < x_2 < \dots < x_{n-1} < G$ be a maximal chain of modular flats of G. Let a_i be the number of points in x_i but not in x_{i-1} for $i = 2, 3, \dots, n$. Then the modular factorization theorem [5] implies that $p(G;\lambda) = (\lambda - 1)(\lambda - a_2)(\lambda - a_3)\dots(\lambda - a_n)$. By Lemma 2.2, we have $a_i \le a_{i+1}$ for each $i = 1, 2, \dots, n-1$. Thus we can conclude that $a_i = q^{i-1}$ for $i = 2, 3, \dots, n$.

We prove this theorem by induction on n. For $n = 1$ and $n = 2$, the theorem is true. Assume it holds for a geometry of rank less than n. Let a be a point in G. Then G/a is upper homogeneous and has a modular copoint and $p(G/a;\lambda) = p(x_{n-1};\lambda) = (\lambda - 1)\,(\lambda - q)(\lambda - q^2)\dots(\lambda - q^{n-2})$. By the induction hypothesis, $G/a \cong PG(n-2, q)$ for every point a in G.

Since projective geometries are modular, Proposition 2.4 implies that $W(s,1)$ is the same as the number of points in $PG(s-1, q)$. Thus $W(s,1) = \dfrac{q^s - 1}{q - 1} = \dbinom{s}{1}_q$ for $s = 1, 2, \dots, n-1$. By Theorem 2.5, we have

$$\sum_s w(n,s)W(s,t) = \delta(n,t).$$

Let $t = 1$ and $n > 1$. Then we have

$$\begin{aligned} W(n,1) &= -\sum_{s=1}^{n-1} w(n,s)W(s,1) \\ &= -\sum_{s=0}^{n-1}(-1)^{n-s}q^{\binom{n-s}{2}}\binom{n}{s}_q\binom{s}{1}_q \\ &= -\sum_{s=0}^{n}(-1)^{n-s}q^{\binom{n-s}{2}}\binom{n}{s}_q\binom{s}{1}_q + \binom{n}{1}_q = \binom{n}{1}_q \\ &= W(n,n-1). \end{aligned}$$

Thus Proposition 2.4 implies that G is modular. Also Lemma 2.1 implies that G is non-splitting and so G is connected. Since G is a connected modular geometry, by Proposition 2.3, we can conclude that G is isomorphic to a projective geometry. Therefore $G \cong PG(n-1,q)$.

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