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A NONEXISTENCE THEOREM FOR STABLE EXPONENTIALLY HARMONIC MAPS

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1. Introduction

Let M and N be compact Riemannian manifolds and $f : M \to N$ be a smooth map. Following J. Eells, f is *exponentially harmonic* if it represents a critical point of the exponential energy integral

$$\mathbf{E}(f) = \int_M \exp(\|df\|^2)\, dM$$

where $\|df\|^2$ is the energy density defined as $\sum_{i=1}^{m} \|df(e_i)\|^2, m = \dim M$, for orthonormal frame e_i of M. The Euler-Lagrange equation of the exponential energy functional $\mathbf{E}$ can be written

$$\exp(\|df\|^2)(\tau(f) + df(\nabla\|df\|^2)) = 0$$

where $\tau(f)$ is the tension field along f. Hence, if the energy density is constant, every harmonic map is exponentially harmonic and vice versa.

An exponentially harmonic map is called *stable* if it represents, furthermore, a local minimum point of the exponential energy. When the target manifold is the standard sphere S^n, it is well known that every stable harmonic map $f : M \to S^n, n \geq 3$, is constant [2]. This is not the case with exponentially harmonic maps since every identity map of M is a stable exponentially harmonic map [1]. In this note, however, we can prove the following

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THEOREM. *Let M be a compact Riemannian manifold of dimension m. Every nonconstant exponentially harmonic map $f : M \to S^n$ is unstable if $\|df\|^2(x) < n-2$ for every $x \in M$.*

2. Proof

We begin with a property of conformal vector fields on S^n. Embed S^n canonically into $\mathbf{R}^{n+1}$. Let $\phi : \mathbf{R}^{n+1} \to \mathbf{R}$ be a linear map with $\phi(O) = O$, $\|\nabla\phi\| = 1$. Set $V_\phi = \nabla\phi$ and let V_ϕ^t and V_ϕ^n be tangential and normal components of V_ϕ, respectively and ν be the outward unit normal vector to S^n, then we have the following lemma. For proof, refer, for example, to [2].

LEMMA. *For any vector X tangent to S^n,*

$$\nabla_X V_\phi^t = -\langle V_\phi, \nu\rangle X.$$

Let $f : M \to S^n$ be a nonconstant exponentially harmonic map. Consider V_ϕ^t and the variation of f in the direction of V_ϕ^t, then by the second variation formula for $\mathbf{E}$ (cf. [1]) we have

$$\begin{aligned}\frac{d^2\mathbf{E}_\phi}{dt^2}(0) = \int_M \exp(\|df\|^2) \sum_{i=1}^m (\langle \nabla_{df(e_i)} V_\phi^t, df(e_i)\rangle^2 \\ + \|\nabla_{df(e_i)} V_\phi^t\|^2 - \langle R(df(e_i), V_\phi^t) df(e_i), V_\phi^t\rangle)\, dM\end{aligned}$$

where R is the curvature tensor of S^n. Write $\frac{d^2\mathbf{E}_\phi}{dt^2}(0)$ as

$$\frac{d^2\mathbf{E}_\phi}{dt^2}(0) = \int_M \exp(\|df\|^2)\left(\sum_{i=1}^m ((\mathbf{A}) + (\mathbf{B}) - (\mathbf{C}))\right) dM$$

and now calculate. By Lemma,

$$\|\nabla_{df(e_i)} V_\phi^t\|^2 = \langle V_\phi, \nu\rangle^2 \|df(e_i)\|^2$$

and since

$$\langle R(df(e_i), V_\phi^t) df(e_i), V_\phi^t\rangle = \|df(e_i)\|^2 \|V_\phi^t\|^2 - \langle df(e_i), V_\phi^t\rangle^2$$

we have

$$\sum_{i=1}^{m}((\mathbf{B})-(\mathbf{C}))$$
$$=\sum_{i=1}^{m}\left(\|df(e_i)\|^2(\langle V_\phi^t,\nu\rangle^2-\|V_\phi^t\|^2)+\langle df(e_i),V_\phi^t\rangle^2\right)$$
$$:=\sum_{i=1}^{m}(\mathbf{D}).$$

Consider now $n+1$ linear functions ϕ_j such that $V_{\phi_j}^t := V_j$ form an orthonormal basis of $\mathbf{R}^{n+1}$ and calculate $\sum_{j=1}^{n+1}\frac{d^2\mathbf{E}_{\phi_j}}{dt^2}(0)$. From

$$\sum_j\langle df(e_i),V_j^t\rangle^2=\sum_j\langle df(e_i),V_j\rangle^2=\|df(e_i)\|^2,$$
$$\sum_j\langle V_j,\nu\rangle^2=\|\nu\|^2=1,$$
$$\sum_j(\langle V_j,\nu\rangle^2-\|V_j^t\|^2)=\sum_j(\|V_j^n\|^2-\|V_j^t\|^2)$$
$$=\sum(2\|V_j^n\|^2-\|V_j\|^2)=2-(n+1)$$

we have

$$\sum_{j=1}^{n+1}\int_M\exp(\|df\|^2)\sum_{i=1}^{m}(\mathbf{D})\,dM=(2-n)\int_M\|df\|^2\exp(\|df\|^2)\,dM,$$

and if $\|df\|^2<n-2$, we have

$$\sum_{j=1}^{n+1}\sum_{i=1}^{m}\langle\nabla_{df(e_i)}V_{\phi_j}^t,df(e_i)\rangle^2=\sum_j\sum_i\langle V_{\phi_j},\nu\rangle^2\|df(e_i)\|^4$$
$$=\sum_i\|df(e_i)\|^4\sum_j\langle V_{\phi_j},\nu\rangle^2=\sum_i\|df(e_i)\|^4$$
$$\le\left(\sum_i\|df(e_i)\|^2\right)^2=\|df\|^4<(n-2)\|df\|^2$$

and consequently we have

$$\sum_{j=1}^{n+1} \int_M \exp(\|df\|^2) \sum_{i=1}^{m} (\mathbf{A})\, dM < (n-2) \int_M \|df\|^2 \exp(\|df\|^2)\, dM.$$

Hence, we have finally,

$$\sum_{j=1}^{n+1} \frac{d^2 \mathbf{E}_{\phi_j}}{dt^2}(0) = \sum_{j=1}^{n+1} \int_M \exp(\|df\|^2) \left(\sum_{i=1}^{m} ((\mathbf{A}) + (\mathbf{B}) - (\mathbf{C})) \right) dM < 0.$$

Therefore, at least one $\frac{d^2 \mathbf{E}_{\phi_j}}{dt^2}(0)$ should be negative, that is, a non-constant exponentially harmonic map f with $\|df\|^2 < n-2$ is not stable. This completes the proof.

References

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