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A PROPERTY OF CERTAIN ANALYTIC FUNCTIONS

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1. Introduction

Let N be the class of functions of the form

(1.1) $$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. If $p(z) \in N$ satisfies $\mathrm{Re}\, p(z) > 0$ $(z \in U)$, then $p(z)$ is called a Carathéodory function (cf. Goodman [2]).

Let A denote the class of functions of the form

(1.2) $$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

A function $f(z) \in A$ is said to be a member of S_k^* if it satisfies

(1.3) $$\mathrm{Re}\left\{\left(\frac{zf'(z)}{f(z)}\right)^k\right\} > 0 \qquad (z \in U)$$

for some $k > 0$. A function $f(z)$ in S_1^* is said to be starlike in U. A function $f(z) \in A$ is said to be a member of the class T_k if it satisfies

(1.4) $$\mathrm{Re}\left\{\left(\frac{f(z)}{z}\right)^k\right\} > 0 \qquad (z \in U)$$

for some $k > 0$. Further, let R_k be the subclass of A consisting of functions $f(z)$ which satisfy

(1.5n) $$\mathrm{Re}\left\{(f'(z))^k\right\} > 0 \qquad (z \in U)$$

for some $k > 0$. A function $f(z)$ in R_1 is said to be close-to-convex in U (cf. Duren[1]).

For $p(z) \in N$, Nunokawa [5] has shown that

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THEOREM A. *If* $p(z) \in \mathrm{N}$ *satisfies*

$$\left|\mathrm{Im}\frac{zp'(z)}{p(z)}\right| < 1 \qquad (z \in \mathrm{U}) \tag{1.6}$$

then $\mathrm{Re}p(z) > 0 \quad (z \in \mathrm{U})$.

2. Main Result

To derive our main result, we have to recall here the following lemma due to Nunokawa [5].

LEMMA. *Let* $p(z)$ *be in the class* N. *If there exist a point* z_0 *in* U *such that* $\mathrm{Re}p(z) > 0 \quad (|z| < |z_0|)$, $\mathrm{Re}p(z_0) = 0$ *and* $p(z_0) \neq 0$, *then*

$$p(z_0) = ia \quad (a \neq 0)$$

and

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\frac{k}{2}\left(a + \frac{1}{a}\right),$$

where k *is real and* $k \geq 1$.

REMARK. The above lemma is the generalization of Jack's lemma by Jack [3] (also, by Miller and Mocanu [4]).

Applying the above lemma, we have

THEOREM. *If* $p(z) \in \mathrm{N}$ *satisfies*

$$\frac{zp'(z)}{p(z)} \neq i\alpha \qquad (z \in \mathrm{U}), \tag{2.1}$$

then $\mathrm{Re}p(z) > 0 \quad (z \in \mathrm{U})$, *where* α *is real and* $|\alpha| \geq 1$.

Proof. Suppose that the function $p(z)$ has a zero of order n at a point $z = \beta$. Then we can write

$$p(z) = (z - \beta)^n q(z), \tag{2.2}$$

where $q(z)$ is analytic in U and $q(\beta) \neq 0$. It follows from (2.2) that

$$\frac{zp'(z)}{p(z)} = \frac{nz}{z-\beta} + \frac{zq'(z)}{q(z)}. \tag{2.3}$$

The imaginary part of the right-hand side of (2.3) can take any values when z tends to β. This contradicts our condition (2.1). Therefore, we have $p(z) \neq 0$ for all $z \in$ U.

Next, if there exists a point $z_0 \in$ U such that $\mathrm{Re}\,p(z) > 0 \quad (|z| < |z_0|)$, $\mathrm{Re}\,p(z_0) = 0$, and $p(z_0) \neq 0$, then Lemma leads us to $p(z_0) = ia \quad (a \neq 0)$ and

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\frac{k}{2}\left(a + \frac{1}{a}\right) \quad (k \geq 1). \tag{2.4}$$

If $a > 0$, then

$$\mathrm{Im}\frac{z_0 p'(z_0)}{p(z_0)} = \frac{k}{2}\left(a + \frac{1}{a}\right) \geq 1, \tag{2.5}$$

and if $a < 0$, then

$$\mathrm{Im}\frac{z_0 p'(z_0)}{p(z_0)} = -\frac{k}{2}\left(|a| + \frac{1}{|a|}\right) \leq -1. \tag{2.6}$$

This shows that

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\alpha \quad (|\alpha| \geq 1). \tag{2.7}$$

We note that (2.7) contradicts our condition (2.1). Thus we conclude that $\mathrm{Re}\,p(z) > 0$ for all $z \in$ U.

REMARK. It is easy to see that Theorem is an improvement of Theorem A by Nunokawa [5]

Letting $p(z) = (zf'(z)/f(z))^k$ in Theorem, we have

COROLLARY 1. *If* $f(z) \in \mathrm{A}$ *satisfies*

$$\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \neq -1 + i\alpha \qquad (z \in \mathrm{U}), \tag{2.8}$$

then $f(z) \in \mathrm{S}_k^*$, *where* α *is real and* $|\alpha| \geq k > 0$.

Making $p(z) = (f(z)/z)^k$ in Theorem, we have

COROLLARY 2. *If* $f(z) \in \mathrm{A}$ *satisfies*

$$\frac{zf'(z)}{f(z)} \neq 1 + i\alpha \qquad (z \in \mathrm{U}), \tag{2.9}$$

then $f(z) \in \mathrm{T}_k$, *where* α *is real and* $|\alpha| \geq k > 0$.

Finally, taking $p(z) = (f'(z))^k$ in Theorem, we have

COROLLARY 3. *If* $f(z) \in \mathrm{A}$ *satisfies*

$$\frac{zf''(z)}{f'(z)} \neq i\alpha \quad (z \in \mathrm{U}), \tag{2.10}$$

then $f(z) \in \mathrm{R}_k$, *where* α *is real and* $|\alpha| \geq k > 0$.

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