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SOME DECOMPOSITION OF MODULAR $sp_4(F)$–MODULES USING DIMENSION FORMULA

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1. Introduction

Simple Lie algebras over algebraically closed field with characteristic $p > 7$ were classified by H. Strade and R.L. Wilson in 1991 [7]. All modular representations of simple Lie algebras, however, are not classified although some restricted modular representations have been done earlier by Curtis and Steinberg. In connection with this, we would like to decompose basic $sp_4(F)$-modules $sl_4(F)$ and $gl_4(F)$ over nonzero characteristic by way of characteristic zero case.

The format of this paper runs as follows : Preliminary results and notations are introduced in §2. Dimensions of ordinary $sp_4(F)$ irreducible modules are obtained in §3 followed by ordinary decomposition of $sp_4(F)$-modules $sl_4(F)$ and $gl_4(F)$ in §4. Finally modular decomposition of $sp_4(F)$-modules in §5 and some facts about modular irreducible $sp_4(F)$-modules in §6 finish this paper. Special thanks are due to H. Strade for his helpful suggestion.

2. Preliminaries

The Lie algebras of type C_l are none other than symplectic Lie algebras which are constructed on even dimensional vector spaces, the vector space being considered over an algebraically closed field F of characteristic zero. Since $sp_2(F) = sl_2(F)$, the irreducible $sp_2(F)$-modules are nothing but $sl_2(F)$- irreducible modules [5]. So their dimensions are easily recognized. Next, there arises a natural question

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what the dimensions of $sp_4(F)$- irreducible modules are. This may be solved through the dimension formula [1].

Let the dimension of $V(\lambda)$ for $\lambda \in \Lambda^+$ be denoted by $\deg(\lambda)$, where Λ^+ denotes the set of dominant integral linear functions. Since $V(\lambda)$ is a direct sum of weight spaces, we see that

$$\deg(\lambda) = \sum_{\mu \in \pi(\lambda)} m_\lambda(\mu)$$

where $m_\lambda(\mu) = \dim V(\lambda)_\mu$ for an integral linear function $\mu \in H^*$. By virtue of this and Weyl's formula, we obtain

PROPOSITION 2.1. *For $\lambda \in \Lambda^+$, we obtain*

$$\deg(\lambda) = \prod_{\alpha \succ 0} (\lambda + \delta, \alpha) / \prod_{\alpha \succ 0} (\delta, \alpha)$$

where $\delta = \frac{1}{2} \sum_{\beta \succ 0} \beta$ and β ranges over all positive roots.

Proof. See Chap. VI [1] for notations and proof. ❑

3. Dimensions of $sp_4(F)$-irreducible modules

Let $h_1 := \mathrm{diag}(1, 1, -1, -1)$, $h_2 := \mathrm{diag}(1, -1, -1, 1)$ and $H = Fh_1 \oplus Fh_2$. Since H becomes a Cartan subalgebra of $sp_4(F)$, we have a Cartan decomposition

$$L = H \oplus \coprod_{(i,j)} L_{\alpha(i,j)}$$

where $(i, j) \neq (0, 0)$ runs over $\{ -2, 0, 2 \}^2$ and $\alpha(i, j)$ denotes the roots for the elementary basis elements of $sp_4(F)$, i. e.,

$$h_1,\ h_2,\ x_1 = E_{13},\ x_2 = E_{24},\ x_3 = E_{14} + E_{23},\ x_4 = E_{12} - E_{43}$$

and their transposes. Explicitly $\alpha(i, j)$ denotes the linear form α satisfying $\alpha(h_1) = i$, $\alpha(h_2) = j$ and E_{ij} indicates an elementary matrix whose (i,j)-th component is 1 with all others zero. There is a root system Φ consisting of 8 roots

$$\alpha(2,2),\quad \alpha(-2,-2),\quad \alpha(2,-2),\quad \alpha(-2,2),$$
$$\alpha(2,0),\quad \alpha(-2,0),\quad \alpha(0,2),\quad \alpha(0,-2).$$

Select a basis Δ of Φ as $\Delta = \{\alpha(2,2), \alpha(0,-2)\}$. Put $\alpha_1 := \alpha(0,-2)$ and $\alpha_2 := \alpha(2,2)$. Let the fundamental dominant weights be $\{\lambda_1, \lambda_2\}$ relative to Δ. Then $\lambda_1 = \alpha_1 + 2^{-1}\alpha_2$, and $\lambda_2 = \alpha_1 + \alpha_2$.

Now recall

$$\begin{aligned}\deg(\lambda) &= \prod_{\alpha\succ 0}(\lambda+\delta,\alpha)/\prod_{\alpha\succ 0}(\delta,\alpha)\\ &= \prod_{\alpha\succ 0}\langle\lambda+\delta,\alpha\rangle/\prod_{\alpha\succ 0}\langle\delta,\alpha\rangle\end{aligned}$$

for $\lambda = m_1\lambda_1 + m_2\lambda_2$ with integers $m_i \geq 0$ and for $\delta = 2^{-1}\sum_{\alpha\succ 0}\alpha = \sum\lambda_i$. Here $\langle x,y\rangle := 2(x,y)/(y,y)$ as usual. Also recall $\langle\beta,\alpha\rangle = r-q$ for $\beta - r\alpha, \cdots, \beta + q\alpha$ which is a α-string through β.

In this case $\delta = 2^{-1}(4\alpha_1 + 3\alpha_2)$. So we have $\langle\delta, 2\alpha_1+\alpha_2\rangle = 2$, $\langle\delta,\alpha_1+\alpha_2\rangle = 3$, $\langle\delta,\alpha_i\rangle = 1$ for $i = 1,2$. Hence we obtain

$$\begin{aligned}\prod_{\alpha\succ 0}\langle\delta,\alpha\rangle &= \langle\delta,\alpha_1\rangle\langle\delta,\alpha_2\rangle\langle\delta,\alpha_1+\alpha_2\rangle\langle\delta\ 2\alpha_1+\alpha_2\rangle\\ &= 6\\ \prod_{\alpha\succ 0}\langle\lambda+\delta,\alpha\rangle &= \prod_{\alpha\succ 0}\{\langle\lambda,\alpha\rangle + \langle\delta,\alpha\rangle\}\\ &= (m_1+1)(m_2+1)(m_1+2m_2+3)(m_1+m_2+2).\end{aligned}$$

So we have

$$\deg(\lambda) = (m_1+1)(m_2+1)(m_1+2m_2+3)(m_1+m_2+2)/6.$$

Incidentally, the trivial module F corresponds to $\lambda = 0\lambda_1 + 0\lambda_2 = 0$ and $sp_4(F)$ corresponds to $\lambda = 2\lambda_1 + \lambda_2$. Hence we have proved.

PROPOSITION 3.1. *For $\lambda \in \Lambda^+$ and for a choice base Δ as above, we have*

$$\deg(\lambda) = (m_1+1)(m_2+1)(m_1+2m_2+3)(m_1+m_2+2)/6.$$

4. Ordinary decomposition of some $sp_4(F)$-modules

For an algebraically closed field F of characteristic zero, we may decompose $sp_4(F)$-modules $sl_4(F)$ and $gl_4(F)$ by virtue of weight space decomposition method. We have seen the various dimensions of irreducible $sp_4(F)$-modules over F in the preceding section. Now we shall identify the decomposition of $sl_4(F)$ and $gl_4(F)$ in view of this.

Put $h_3 := \mathrm{diag}(1,-1,1,-1)$, $w_1 := E_{14} - E_{23}$ and $w_2 := E_{12} + E_{43}$. Then we may see easily that $V = sl_4(F)$ is 15-dimensional and has a basis

$$\{ h_1, h_2, h_3, x_1, x_2, x_3, x_4, w_1, w_2, \text{transposes of } x_i, \text{transposes of } w_j \}$$

with $i = 1,2,3$ and $j = 1,2$.

We may also see easily that these basis elements of V have weights with respect to $\{ h_1, h_2 \}$ as follows :

x_1 has a weight $\alpha(2,2)$, $\quad {}^tx_1$ has a weight $\alpha(-2,-2)$

x_2 has a weight $\alpha(2,-2)$, $\quad {}^tx_2$ has a weight $\alpha(-2,2)$

x_3 has a weight $\alpha(2,0)$, $\quad {}^tx_3$ has a weight $\alpha(-2,0)$

x_4 has a weight $\alpha(0,2)$, $\quad {}^tx_4$ has a weight $\alpha(0,-2)$

w_1 has a weight $\alpha(2,0)$, $\quad {}^tw_1$ has a weight $\alpha(-2,0)$

w_2 has a weight $\alpha(0,2)$, $\quad {}^tw_2$ has a weight $\alpha(0,-2)$

Note that $\{ h_1, h_2, h_3 \}$ has a weight $\alpha(0,0)$, where $\alpha(i,j)$ denotes a linear form such that $\alpha(h_1) = i$ and $\alpha(h_2) = j$. The $sp_4(F)$-module $V = sl_4(F)$ has a weight space decomposition as

$$\begin{aligned} V &= V_0 \oplus \coprod_{\substack{(i,j)\neq 0 \\ (i,j)\in\{-2,0,2\}^2}} V_{\alpha(i,j)} \\ &= V_0 \oplus V_{\alpha(2,2)} \oplus V_{\alpha(-2,-2)} \oplus V_{\alpha(2,-2)} \oplus V_{\alpha(-2,2)} \\ &\quad \oplus V_{\alpha(2,0)} \oplus V_{\alpha(-2,0)} \oplus V_{\alpha(0,2)} \oplus V_{\alpha(0,-2)}, \end{aligned}$$

where $\dim V_{\alpha(2,-2)} = \dim V_{\alpha(-2,2)} = 1$ and $\dim V_{\alpha(2,0)} = \dim V_{\alpha(-2,0)} = \dim V_{\alpha(0,2)} = \dim V_{\alpha(0,-2)} = 2$. Moreover $V = sp_4(F) \oplus \mathcal{U}(L)w_1$, where $L = sp_4(F)$ has a maximal vector of weight $\alpha(2,-2)$ in $L_{\alpha(2,-2)} = Fx_2$ and $\mathcal{U}(L)w_1$ has a maximal vector of weight $\lambda(2,0)$ in Fw_1 with the universal enveloping algebra $\mathcal{U}(L)$ of L. Here $\mathcal{U}(L)w_1$ is the same as $V(\lambda(2,0)) \cong V(\lambda)$ for $\lambda = 0\lambda_1 + \lambda_2$ since there exists only one $sp_4(F)$-irreducible module of dimension 5 by proposition (3.1).

All we have to check is that

$$\deg \lambda(2,0) := \dim V(\lambda(2,0)) = \dim \mathcal{U}(L)w_1 = 5.$$

Since $\lambda_1 = \alpha_1 + 2^{-1}\alpha_2$ and $\lambda_2 = \alpha_1 + \alpha_2$, we see that

$$\begin{aligned}\lambda(2,0) &= m_1\lambda_1 + m_2\lambda_2 \\ &= m_1(\alpha_1 + 2^{-1}\alpha_2) + m_2(\alpha_1 + \alpha_2) \\ &= (m_1 + m_2)\alpha(0,-2) + (2^{-1}m_1 + m_2)\alpha(2,2)\end{aligned}$$

means $m_1 = 0$ and $m_2 = 1$, where $\alpha_1 = \alpha(0,-2)$ and $\alpha_2 = \alpha(2,2)$. If we select Δ as in §3, we obtain

$$\prod_{\alpha \succ 0} \langle \delta, \alpha \rangle = 6$$

and

$$\begin{aligned}&\prod_{\alpha \succ 0} \langle \lambda + \delta, \alpha \rangle \\ &= \prod_{\alpha \succ 0} \{\langle \lambda, \alpha \rangle + \langle \delta, \alpha \rangle\} \\ &= (\langle \lambda_2, \alpha_1 \rangle + \langle \delta, \alpha_1 \rangle)(\langle \lambda_2, \alpha_2 \rangle + \langle \delta, \alpha_2 \rangle) \\ &\quad + (\langle \lambda_2, \alpha_1 + \alpha_2 \rangle + \langle \delta, \alpha_1 + \alpha_2 \rangle)(\langle \lambda_2, 2\alpha_1 + \alpha_2 \rangle + \langle \delta, 2\alpha_1 + \alpha_2 \rangle) \\ &= 30.\end{aligned}$$

So we obtain

$$\begin{aligned}\deg \lambda(2,0) &= \dim V(\lambda) \\ &= \prod_{\alpha \succ 0} \langle \lambda + \delta, \alpha \rangle / \prod_{\alpha \succ 0} \langle \delta, \alpha \rangle \\ &= 30/6 = 5.\end{aligned}$$

Hence we have

$$V(\lambda_2) = Fh_3 \oplus Fw_1 \oplus F^t w_1 \oplus Fw_2 \oplus F^t w_2$$

as a by-product. Obviously we obtain

$$gl_4(F) = sl_4(F) \oplus FI_4$$

with I_4 the 4×4 identity matrix. So we have proven

PROPOSITION 4.1. *Let F be an algebraically closed field of characterictic zero. Then $sp_4(F)$-modules $sl_4(F)$ and $gl_4(F)$ have their decompositions*

$$sl_4(F) = sp_4(F) \oplus \mathcal{U}(L)w_1 \quad \text{and}$$
$$gl_4(F) = sp_4(F) \oplus \mathcal{U}(L)w_1 \oplus FI_4,$$

where $\mathcal{U}(L)$ denotes the universal enveloping algebra of $L = sp_4(F)$ and it turns out that

$$\mathcal{U}(L)w_1 = Fh_3 \oplus Fw_1 \oplus F^t w_1 \oplus Fw_2 \oplus F^t w_2.$$

Moreover $sp_4(F) \cong V(2\lambda_1 + \lambda_2)$ and $\mathcal{U}(L)w_1 = V(\lambda_2)$ in terms of notations of §2.

5. Modular decomposition of $sl_4(F)$ and $gl_4(F)$

We have seen that when F is an algebraically closed field of characteristic zero the natural $sp_4(F)$-modules $sl_4(F)$ and $gl_4(F)$ are decomposed into irreducible pieces explicitly. The similar results may be conjectured when F is any field of characteristic p. But we must clarify this conjecture in reality. The two natural $sp_4(F)$-modules $sl_4(F)$ and $gl_4(F)$ have their decompositions as

$$sl_4(F) = sp_4(F) \oplus \mathcal{U}(L)w_1 \quad \text{and}$$
$$gl_4(F) = sp_4(F) \oplus \mathcal{U}(L)w_1 \oplus FI_4,$$

and it turns out that $\mathcal{U}(L)w_1 = Fh_3 \oplus Fw_1 \oplus F^t w_1 \oplus Fw_2 \oplus F^t w_2$, where F is an algebraically closed field of characteristic 0.

The question is "what about F of characteristic $p > 0$?" Now let

$$V(\lambda(0,2))_F = Fh_3 \oplus Fw_1 \oplus F^t w_1 \oplus Fw_2 \oplus F^t w_2$$

and let

$$V(\lambda(0,2))_{\mathbb{Z}} = \mathbb{Z}h_3 \oplus \mathbb{Z}w_1 \oplus \mathbb{Z}^t w_1 \oplus \mathbb{Z}w_2 \oplus \mathbb{Z}^t w_2,$$

so $V(\lambda(0,2))_F = V(\lambda(0,2))_{\mathbb{Z}} \otimes_{\mathbb{Z}} F$ whose notations are relevant to those in §4. So we know easily that $sl_4(\mathbb{Z}) = sp_4(\mathbb{Z}) \oplus V(\lambda(0,2))_{\mathbb{Z}}$. Hence

$$\begin{aligned} sl_4(F) &= sl_4(\mathbb{Z}) \otimes_{\mathbb{Z}} F \\ &= \{ sp_4(\mathbb{Z}) \otimes_{\mathbb{Z}} F \} \oplus \{ V(\lambda(0,2))_{\mathbb{Z}} \otimes_{\mathbb{Z}} F \} \\ &= sp_4(F) \oplus V(\lambda(0,2))_F \end{aligned}$$

as an $sp_4(F)$-irreducible decomposition for $p \neq 2$ since any elementary basis element in $V(\lambda(0,2))_F$ generates the $sp_4(F)$-module $V(\lambda(2,0))_F$ itself. Similarly we obtain

$$gl_4(\mathbb{Z}) \otimes_{\mathbb{Z}} F = sp_4(F) \oplus V(\lambda(0,2))_F \oplus FI_4.$$

So we have proved

PROPOSITION 5.1. *The decomposition of the natural $sp_4(F)$-modules $sl_4(F)$ and $gl_4(F)$ over any field F with $p \neq 2$ is as follows :*

$$\begin{aligned} sl_4(F) &= sp_4(F) \oplus V(\lambda(0,2))_F \\ gl_4(F) &= sp_4(F) \oplus V(\lambda(0,2))_F \oplus FI_4. \end{aligned}$$

6. Some facts about modular irreducible $sp_4(F)$-modules

It is not odd to conjecture that all finite dimensional irreducible $sp_4(F)$-modules over an algebraically closed field F of characteristic $p > 0$ may have a common upper bound of their dimensions. This section answers the question in this conjecture in the affirmative in addition to stating that $sp_4(F)$ is a restricted Lie algebra.

We put $\bar{h}_1 := \mathrm{diag}(1,0,-1,0)$ and $\bar{h}_2 := \mathrm{diag}(0,1,0,-1)$. Let x_1, ${}^t x_1$, x_2, ${}^t x_2$, x_3, ${}^t x_3$, x_4, ${}^t x_4$ denote the elementary basis elements as in §3. Then we see immediately that for $L = sp_4(F)$,

$$\begin{array}{llll}(\mathrm{ad}\, x_1)^p = 0, & (\mathrm{ad}\, {}^t x_1)^p = 0, & (\mathrm{ad}\, x_2)^p = 0, & (\mathrm{ad}\, {}^t x_2)^p = 0 \\ (\mathrm{ad}\, x_3)^p = 0, & (\mathrm{ad}\, {}^t x_3)^p = 0, & (\mathrm{ad}\, x_4)^p = 0, & (\mathrm{ad}\, {}^t x_4)^p = 0,\end{array}$$

and

$$(\mathrm{ad}\, \bar{h}_1)^p - \mathrm{ad}\, \bar{h}_1 = 0, \qquad (\mathrm{ad}\, \bar{h}_2)^p - \mathrm{ad}\, \bar{h}_2 = 0$$

since $p|(2^{p-1} - 1)$. We shall denote the canonical basis

$$\{ \bar{h}_1, \bar{h}_2, x_1, {}^t x_1, x_2, {}^t x_2, x_3, {}^t x_3, x_4, {}^t x_4 \}$$

by $\{ u_i : i = 1, 2, \cdots, 10 \}$ in any order prescribed, and denote the universal enveloping algebra of L by $\mathcal{U} := \mathcal{U}(L)$.

The PBW theorem asserts that the standard monomials $u_{i_1}^{k_1} u_{i_2}^{k_2} \cdots u_{i_{10}}^{k_{10}}$ with $i_1 < i_2 < \cdots < i_{10}$ and with $k_j \geq 0$ constitute a basis for $\mathcal{U}$. There is an obvious filtration of $\mathcal{U}$ defined by

$$\mathcal{U}^{(k)} = F \cdot 1 \oplus L \oplus L^2 \oplus \cdots \oplus L^k.$$

Evidently the monomials $u_{i_1}^{k_1} u_{i_2}^{k_2} \cdots u_{i_{10}}^{k_{10}}$ with $k_1 + k_2 + \cdots + k_{10} \leq k$ form a basis for $\mathcal{U}^{(k)}$. Since $(\mathrm{ad}\, u_i)^p = \mathrm{ad}\, u_i^{[p]}$ for some $u_i^{[p]} \in L = sp_4(F)$ by the above remark, $z_i := u_i^p - u_i^{[p]}$ commutes with every $l \in L$, and so $z_i \in$ Center of $\mathcal{U}$. Hence $u_i^{[p]}\ (\in L) = u_i^p - z_i$ belongs to $\mathcal{U}^{(p-1)}$.

PROPOSITION 6.1. *The elements of the form*

$$z_{i_1}^{\sigma_1} z_{i_2}^{\sigma_2} \cdots z_{i_{10}}^{\sigma_{10}} u_{i_1}^{\lambda_1} u_{i_2}^{\lambda_2} \cdots u_{i_{10}}^{\lambda_{10}} \quad \textit{with}$$
$$i_1 < i_2 < \cdots < i_{10}, \ \ \sigma_j \geq 0, \ \ 0 \leq \lambda_j < p$$

constitute a basis for $\mathcal{U}$.

Proof. See §7, Chap V in [2]. □

PROPOSITION 6.2. *Suppose that L is $sp_4(F)$ over a field F as before. Then we have a unique mapping $a \mapsto a^{[p]}$ of L into L satisfying that $(\mathrm{ad}\, u_i)^p = \mathrm{ad}\, u_i^{[p]}$ as above and that L is a restricted Lie algebra with respect to the p-mapping $a \mapsto a^{[p]}$.*

Proof. See also §7, Chap V in [2]. □

We see from Lemma (6.1) that the set of all elements

$$z_{i_1}^{\sigma_1} z_{i_2}^{\sigma_2} \cdots z_{i_{10}}^{\sigma_{10}} u_{i_1}^{\lambda_1} u_{i_2}^{\lambda_2} \cdots u_{i_{10}}^{\lambda_{10}} \qquad \text{with}$$
$$i_1 < i_2 < \cdots < i_{10}, \ \ \sigma_j \geq 0, \ \ 0 \leq \lambda_j < p-1$$

form a basis for $\mathcal{U}(L), L = sp_4(F)$.

Now any irreducible L-module M is of the form $\mathcal{U}(L) \cdot v$ for any $v \in M$. But then as $\mathcal{U}(L)$-modules and so as F-modules, we have $\mathcal{U}(L)/I \cong \mathcal{U}(L) \cdot v$ for some maximal left ideal I of $\mathcal{U}(L)$. If M is finite dimensional, then I is some infinite dimensional maximal left ideal of $\mathcal{U}(L)$ in particular.

By the way, for any element $z \in$ Center of $\mathcal{U}(L)$, we have a $\mathcal{U}(L)$-module homomorphism $z : \mathcal{U}(L) \cdot v \to \mathcal{U}(L) \cdot v$ given by $z(xv) = zxv$ for any element $x \in \mathcal{U}(L)$. Suppose that $\mathcal{U}(L) \cdot v$ is any finite dimensional irreducible L-module. Then

$$\mathcal{U}(L) \cdot v = z \cdot \mathcal{U}(L) \cdot v = \mathcal{U}(L) \cdot zv,$$

which forces z to be a constant by Schur's Lemma. So in this case, $\mathcal{U}(L) \cdot v$ has a basis consisting of at most p^{10} elements. Hence we have the following

PROPOSITION 6.3. *Any finite dimensional irreducible L-module over an algebraically closed field of characteristic $p > 0$ has its F-dimension not greater than p^{10}.*

Recently we have found that any finite dimensional irreducible $sp_4(F)$ -module over F as in the proposition has its dimension not greater than p^4 and that actually there exists an irreducible $sp_4(F)$-module of maximum dimension p^4. We want to reveal this fac in other journals sooner or later.

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