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LOWER BOUNDS OF THE NUMBER OF JUMP OPTIMAL LINEAR EXTENSIONS : PRODUCTS OF SOME POSETS

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1. Introduction

Let P be a finite poset and let $|P|$ be the *number of vertices* in P. A *subposet* of P is a subset of P with the induced order. A *chain* C in P is a subposet of P which is a linear order. The *length* of the chain C is $|C| - 1$. A *linear extension* of a poset P is a linear order $L = x_1, x_2, \ldots, x_n$ of the elements of P such that $x_i < x_j$ in P implies $i < j$. Let $\mathcal{L}(P)$ be the set of all linear extensions of P. E. Szpilrajn [5] showed that $\mathcal{L}(P)$ is not empty.

Let P, Q be two disjoint posets. The *disjoint sum* $P + Q$ of P and Q is the poset on $P \cup Q$ such that $x < y$ if and only if $x, y \in P$ and $x < y$ in P or $x, y \in Q$ and $x < y$ in Q. The *linear sum* $P \oplus Q$ of P and Q is obtained from $P + Q$ by adding the relation $x < y$ for all $x \in P$ and $y \in Q$.

Throughout this section, L denotes an arbitrary linear extension of P. Let $a, b \in P$ with $a < b$. Then b *covers* a denoted $a \prec b$, provided that for any $c \in P$, $a < c \leq b$ implies that $c = b$. A (P, L)*-chain* is a maximal sequence of elements $z_1, z_2, \ldots, z_k$ such that $z_1 \prec z_2 \prec \cdots \prec z_k$ in both L and P. Let $c(L)$ be the number of (P, L)-chains in L.

A consecutive pair (x_i, x_{i+1}) of elements in L is a *jump* (or *setup*) of P in L if x_i is not comparable to x_{i+1} in P. The jumps induce a decomposition $L = C_1 \oplus \cdots \oplus C_m$ of L into (P, L)-chains $C_1, \ldots, C_m$ where $m = c(L)$ and $(\max C_i, \min C_{i+1})$ is a jump of P in L for $i = 1, \ldots, m-1$. Let $s(L, P)$ be the number of jumps of P in L and let $s(P)$ be the minimum of $s(L, P)$ over all linear extensions L of P. The

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number $s(P)$ is called the *jump number* of P. If $s(L,P) = s(P)$ then L is called a *(jump) optimal linear extension* of P. The *width* $\omega(P)$ of P is the maximal number of elements of an *antichain* (mutually incomparable elements) of P. We denote the set of all optimal linear extensions of P by $\mathcal{O}(P)$. M. Chein and M. Habib [1] introduced several aspects of jump number. H.C. Jung [2] studied jump number of some classical posets, Young's Lattice and Subspace Lattice.

Let $N = \{x_1 < y_1, x_2 < y_1, x_2 < y_2\}$ be a poset. Then $L_1 = x_1x_2y_1y_2$, $L_2 = x_1x_2y_2y_1$, $L_3 = x_2x_1y_1y_2$, $L_4 = x_2x_1y_2y_1$, $L_5 = x_2y_2x_1y_1$ are all the possible linear extensions. Thus $|\mathcal{L}(N)|$=5. Also, $s(N)$=1 and L_5 is the only one optimal linear extension of N, that is, $|\mathcal{O}(N)|$=1. It is clear that $|\mathcal{O}(P)| \leq |\mathcal{L}(P)|$. In this paper, we are interested in counting optimal linear extensions of some posets.

In section 2, we introduce elementary examples of counting optimal linear extensions. In section 3, we count jump optimal linear extensions of some products of posets.

2. Elementary Examples

In this section we count optimal linear extensions of some elementary posets by direct counting. A k-chain $\underline{k}$ is a chain of length $k-1$.

EXAMPLE 2.1. We can easily get $s(\underline{a_1} + \cdots + \underline{a_n}) = n - 1$. Let $a_{i_1}, \ldots, a_{i_n}$ be any rearrangement of $a_1, \ldots, a_n$. Then any optimal linear extension of $\underline{a_1} + \cdots + \underline{a_n}$ can be expressed as $\underline{a_{i_1}} \oplus \cdots \oplus \underline{a_{i_n}}$. Thus we get $|\mathcal{O}(\underline{a_1} + \cdots + \underline{a_n})| = n!$.

Let $I_m = 1 + \cdots + 1$ (m times). We define a *generalized tower* to be a poset $T_n^m = I_m \oplus \cdots \oplus I_m$ (n times).

EXAMPLE 2.2. Since every linear extension is also optimal linear extension, we get easily $s(T_n^m) = mn - n$ and $|\mathcal{O}(T_n^m)| = (m!)^n$.

An *upward* [*downward*] *rooted tree* T_u [T_d] is a poset whose diagram is an upward [downward] rooted tree.

EXAMPLE 2.3. Let $T = T_u$ or T_d. Then we get $s(T) = \omega(T) - 1$. Let $A = \{x_1, \ldots, x_{\omega(T)}\}$ be maximum size antichain of T. For each $L \in \mathcal{O}(T)$, let L_A be a subposet of L which is a linear extension of A.

Then $L_A \in \mathcal{O}(A)$. Conversely, for each $l \in \mathcal{O}(A)$ there exists a unique $L^{[l]} \in \mathcal{O}(T)$ such that $L^{[l]}_A = l$. Hence $|\mathcal{O}(T)| = |\mathcal{O}(A)| = \omega(T)!$.

Let $K_{m,n} = \{x_i < y_j$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n\}$ be a *complete bipartite poset.*

EXAMPLE 2.4. Since every linear extension L has $m+n-1$ $(K_{m,n}, L)$-chains, $s(K_{m,n}) = m + n - 2$. Now every linear extension of $K_{m,n}$ is optimal linear extension of $K_{m,n}$. Thus we get $|\mathcal{O}(K_{m,n})| = m!n!$.

A ***standard poset on n elements*** S_n is defined to be $\{x_i < y_j$ for $i \neq j$, and $i = 1, \ldots, n$ and $j = 1, \ldots, n\}$.

EXAMPLE 2.5. Since every optimal linear extension L of S_n has exactly two (S_n, L)-chains of length one, $s(S_n) = 2n - 3$. Hence $|\mathcal{O}(S_n)| = \binom{n}{n-2} \cdot (n-2)! \cdot 2 \cdot (n-1)! = n!(n-1)!$.

A *Fences* (or *zigzag*) on n elements is a poset $F_n = \{a_1 < a_2, a_2 > a_3, \ldots\}$.

EXAMPLE 2.6. Note that $s(F_n) = \lceil \frac{n}{2} \rceil - 1$. Let m be a positive integer.
If $n = 2m$, $F_n = \{a_{2i-1} < a_{2i}, a_{2j} > a_{2j+1}$, for $i = 1, \cdots, m$ and $j = 1, \cdots, m-1\}$. Thus $L = \oplus_{i=1}^{m} \{a_{2(m-i)+1}, a_{2(m-i)+2}\}$ is the only one optimal linear extension of F_{2m}.
If $n = 2m+1$, $F_n = \{a_{2i-1} < a_{2i}, a_{2j} > a_{2j+1}$, for $i = 1, \cdots, m$ and $j = 1, \cdots, m\}$. For any $L \in \mathcal{O}(F_n)$, we have

$$L = \{a\} \oplus L' \text{ where } a = a_{2i+1} \text{ for } i = 0, \ldots, m, \text{ and } L' \in \mathcal{O}(F_n \setminus \{a\}).$$

For each $a = a_{2i+1}$, we have $F_n \setminus \{a\} = F_{2i} \bigcup F_{2m-2i}$ where F_{2i} and F_{2m-2i} are disjoint. Thus $|\mathcal{O}(F_n \setminus \{a\})| = \binom{m}{i}$, and $|\mathcal{O}(F_n)| = \sum_{i=0}^{m} \binom{m}{i} = 2^m$.
Hence we get

$$|\mathcal{O}(F_n)| = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2^{(n-1)/2} & \text{otherwise.} \end{cases}$$

For intergers n, k with $n \geq 0$ and $k \geq 0$, the *general crown* S_n^k is the poset of unit length with $n + k$ minimal elements $x_1, \ldots, x_{n+k}$ and

$n+k$ maximal elements $y_1, \dots, y_{n+k}$. The order on S_n^k is defined by $x_i < y_j$ iff $j \notin \{i, i+1, \dots, i+k\}$, where addition is modulo $n+k$. When $k=0$, we get $S_n^0 = S_n$.

EXAMPLE 2.7. Any optimal linear extension L of S_n^k is of the following form:

$$L = D_1 \oplus C_1 \oplus C(k) \oplus C_2 \oplus D_2$$

where D_j is linear sums of $n-2$ one element (S_n^k, L)-chains, and C_j is a two element (S_n^k, L)-chain for $j = 1, 2$, and $C(k)$ is a linear sums of k two element (S_n^k, L)-chains. Then $s(S_n^k) = 2n + k - 3$. Note that for each fixed C_1 there are $(n-2)!$ choices for D_1, and 2^k choices for $C(k)$, and $n-1$ choices for C_2, and $(n-2)!$ choices for D_2. Also, there are $(n+k)(n-1)$ choices for C_1. Hence $|\mathcal{O}(S_n^k)| = 2^k(n+k)\{(n-1)!\}^2$.

3. Main Results

Let P, Q be two posets. The *direct product* $P \times Q$ of P and Q is the poset on $\{(p,q) : p \in P, q \in Q\}$ where $(a,b) \le (c,d)$ if and only if $a \le c$ in P and $b \le d$ in Q. Let P^n be $P \times \cdots \times P$ (n times).

We consider the poset $\underline{a_1} \times \cdots \times \underline{a_n}$ where $a_1, \dots, a_n$ are positive integers. We assume that $a_i \ge 2$ for $i = 1, \dots, n$ and let $a^* = \max\{a_1, \dots, a_n\}$. Without loss of generality, we assume that $a^* = a_1$.

H.C. Jung [3] found jump number of products of chains.

PROPOSITION 3.1.

$$s(\underline{a_1} \times \cdots \times \underline{a_n}) = \left(\prod_{i=1}^{n} a_i\right)/a^* - 1.$$

Let $[k]$ be $\{1, \dots, k\}$. Let G^k be the symmetric group on $[k]$. Define an action $G^k \times \mathcal{O}(\underline{m}^k) \to \mathcal{O}(\underline{m}^k)$ by $(\sigma, L) \mapsto \sigma(L)$ where $\sigma(L)$ is a linear order in $\underline{m}^k$ determined by applying an element $\sigma \in G^k$ to the linear order $L \in \mathcal{O}(\underline{m}^k)$, i.e., if $L = x_1 x_2 \dots x_{m^k}$, then $\sigma(L) = \sigma(x_1)\sigma(x_2)\dots\sigma(x_{m^k})$. For any $(\underline{m}^k, L)$-chain C, $\sigma(C)$ is a $(\underline{m}^k, \sigma(L))$-chain. Thus $\sigma(L)$ is also an optimal linear extension of $\underline{m}^k$, and this action is well-defined.

THEOREM 3.2. *We obtain the lower bound of* $|\mathcal{O}(\underline{m}^n)|$,

$$|\mathcal{O}(\underline{m}^n)| \geq \prod_{k=1}^{n} k^{m^{n-k}}.$$

Proof. Let $\sigma_i = (i, k+1) \in G^{k+1}$ for $i = 1, \dots, k$. For each $y = (y_1, \dots, y_k, p) \in \underline{m}^{k+1}$, define $\sigma_i(y) = (y_1, \dots, y_{i-1}, p, y_{i+1}, \dots, y_k, y_i)$. If $x < y$ in $\underline{m}^{k+1}$, then $\sigma_i(x) < \sigma_i(y)$ in $\underline{m}^{k+1}$. Thus if $L_p = x_1 x_2 \dots x_{m^k}$ $\in \mathcal{O}(\underline{m}^k \times \{p\})$ then

$$\sigma_i(L_p) = \sigma_i(x_1)\sigma_i(x_2)\dots\sigma_i(x_{m^k}) \in \mathcal{O}(\underline{m}^{i-1} \times \{p\} \times \underline{m}^{k-i+1}).$$

Let $\mathcal{O}_o^p = \mathcal{O}(\underline{m}^k \times \{p\})$. Then $|\mathcal{O}(\underline{m}^{i-1} \times \{p\} \times \underline{m}^{k-i+1})| = |\mathcal{O}_o^p| = |\mathcal{O}(\underline{m}^k)|$.
For $L_p \in \mathcal{O}_o^p$ where $p = 1, \dots, m$, let $L = L_1 \oplus L_2 \oplus \cdots \oplus L_m$. By Proposition 3.1, $c(L_p) = m^{k-1}$ for each $p = 1, \dots, m$, so we get $c(L) = m^{k-1} \cdot m = m^k$. Thus Proposition 3.1 implies $L \in \mathcal{O}(\underline{m}^{k+1})$. Let

$$\mathcal{O}_o = \{L : L = L_1 \oplus L_2 \oplus \cdots \oplus L_m \text{ for } L_p \in \mathcal{O}_o^p \text{ where } p = 1, \dots, m\}.$$

Then $\mathcal{O}_o \subseteq \mathcal{O}(\underline{m}^{k+1})$ and $|\mathcal{O}_o| = |\mathcal{O}(\underline{m}^k)|^m$. Now $L \in \mathcal{O}(\underline{m}^{k+1})$ implies $\sigma_i(L) \in \mathcal{O}(\underline{m}^{k+1})$.
For $i = 0, 1, \dots, k$, let $\mathcal{O}_i = \{\sigma_i(L) : L \in \mathcal{O}_o\}$. Then $|\mathcal{O}_i| = |\mathcal{O}(\underline{m}^k)|^m$. But $\mathcal{O}_i \cap \mathcal{O}_j = \emptyset$ for $0 \leq i < j \leq k$. Hence $(k+1)|\mathcal{O}(\underline{m}^k)|^m \leq |\mathcal{O}(\underline{m}^{k+1})|$, and by induction we get $\prod_{k=1}^{n} k^{m^{n-k}} \leq |\mathcal{O}(\underline{m}^n)|$.

In general, equality does not hold. For example, consider $\underline{3}^3 = \{(i,j,k) : i,j,k = 1,2,3\}$. Let $C(i,j) = \{(i,j,l) : l = 1,2,3\}$. Then $L = C(1,1) \oplus C(1,2) \oplus C(2,1) \oplus C(1,3) \oplus C(2,2) \oplus C(3,1) \oplus C(2,3) \oplus C(3,2) \oplus C(3,3)$ is not the form counted in Theorem 3.3. It is clear that $|\mathcal{O}(\underline{m}^n)| \leq |\mathcal{L}(\underline{m}^n)|$.

A *Boolean algebra* B_n *on* n *elements* is $\underline{2}^n$.

COROLLARY 3.3.

$$|\mathcal{O}(B_n)| \geq \prod_{k=1}^{n} k^{2^{n-k}}.$$

THEOREM 3.4. *Let $a_1 = \ldots = a_i > a_{i+1} \geq a_{i+2} \geq \cdots \geq a_n$, then*

$$|\mathcal{O}(\underline{a_1} \times \cdots \times \underline{a_n})| \geq |\mathcal{O}(\underline{a_1} \times \cdots \times \underline{a_i})|^{\prod_{j=i+1}^{n} a_j} |\mathcal{L}(\underline{a_{i+1}} \times \cdots \times \underline{a_n})|.$$

Proof. For any $y \in \underline{a_{i+1}} \times \cdots \times \underline{a_n}$, define $I_y(x) = (x, y)$ where $x \in \underline{a_1} \times \cdots \times \underline{a_i}$. Given $l = z_1 z_2 \ldots z_{a_1 \cdots a_i} \in \mathcal{O}(\underline{a_1} \times \cdots \times \underline{a_i})$, we define $E_y(l) = I_y(z_1) I_y(z_2) \ldots I_y(z_{a_1 \cdots a_i})$. Since $c(l) = c(E_y(l))$, Proposition 3.1 implies $E_y(l) \in \mathcal{O}(\underline{a_1} \times \cdots \times \underline{a_i} \times \{y\})$.
Let $l_2 \in \mathcal{L}(\underline{a_{i+1}} \times \cdots \times \underline{a_n})$. For each $y \in l_2$, choose $l_y \in \mathcal{O}(\underline{a_1} \times \cdots \times \underline{a_i})$. Now we define

$$s(l_2, \{l_y : y \in l_2\}) = (\oplus E_y(l_y) : y \text{ ordered as in } l_2).$$

Since $c(s(l_2, \{l_y : y \in l_2\})) = \prod_{i=2}^{n} a_i$, Proposition 3.1 implies that $s(l_2, \{l_y : y \in l_2\})$ is an optimal linear extension of $\underline{a_1} \times \cdots \times \underline{a_n}$. Let $S(l_2) = \{s(l_2, \{l_y : y \in l_2\})\}$. Then

$$|S(l_2)| = |\mathcal{O}(\underline{a_1} \times \cdots \times \underline{a_i})|^{\prod_{j=i+1}^{n} a_j}.$$

Note that for any $l_2, l_2' \in \mathcal{L}(\underline{a_{i+1}} \times \cdots \times \underline{a_n})$, $S(l_2) \cap S(l_2') = \emptyset$. Now let $\mathcal{O} = \cup S(l_2)$ where $l_2 \in \mathcal{L}(\underline{a_{i+1}} \times \cdots \times \underline{a_n})$. Thus

$$|\mathcal{O}| = |\mathcal{O}(\underline{a_1} \times \cdots \times \underline{a_i})|^{\prod_{j=i+1}^{n} a_j} |\mathcal{L}(\underline{a_{i+1}} \times \cdots \times \underline{a_n})|.$$

Since $|\mathcal{O}(\underline{a_1} \times \cdots \times \underline{a_n})| \geq |\mathcal{O}|$, we get the desired results.

COROLLARY 3.5. *Let $a = a_1 = \ldots = a_i > a_{i+1} \geq a_{i+2} \geq \cdots \geq a_n$, then*

$$|\mathcal{O}(\underline{a_1} \times \cdots \times \underline{a_n})| \geq \left(\prod_{k=1}^{i} k^{a^{i-k}}\right)^{\prod_{j=i+1}^{n} a_j} |\mathcal{L}(\underline{a_{i+1}} \times \cdots \times \underline{a_n})|.$$

In the above Corollary, equality does not hold in general. For example, consider $\underline{4} \times \underline{4} \times \underline{3}$. Let $C_{ij} = \{(l, i, j) : l = 1, 2, 3, 4\}$. We can choose an optimal linear extension $L = C_{11} \oplus C_{12} \oplus C_{21} \oplus \cdots$ which is not counted.

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