Bull. Korean Math. Soc. **32** (1995), No. 2, pp. 153–162

A NON–COMPACT GENERALIZATION OF HORVATH'S INTERSECTION THEOREM*

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1. Introduction

Ky Fan's minimax inequality is an important tool in nonlinear functional analysis and its applications, e.g. game theory and economic theory. Since Fan gave his minimax inequality in [2], various extensions of this interesting result have been obtained (see [4,11] and the references therein). Using Fan's minimax inequality, Ha [6] obtained a non-compact version of Sion's minimax theorem in topological vector spaces, and next Geraghty-Lin [3], Granas-Liu [4], Shih-Tan [11], Simons [12], Lin-Quan [10], Park-Bae-Kang [17], Bae-Kim-Tan [1] further generalize Fan's minimax theorem in more general settings. In [9], using the concept of submaximum, Komiya proved a topological minimax theorem which also generalizes Sion's minimax thoerem and another minimax theorem of Ha in [5] without using linear structures. And next Lin-Quan [10] further generalizes his result to two function versions and non-compact topological settings.

The usual proofs of the classical von Neumann minimax theorem and its various generalizations are based on the deep combinatorial result of Sperner or the Brouwer fixed point theorem. On the other hand, the important role of connectedness in minimax theorems was first noted by Wu [16], followed by Tuy [15] who was able to generalize Sion's minimax theorem. And Joó [8] gave a simple proof of the von Neumann minimax theorem by using the topological concept. Recently, by following the method of Joó, Horvath[7] gave a general result in

Received December 2, 1993.
1991 AMS Subject Classification: Primary 49J35; Secondary 52A07.
Key words and : Minimax theorem, non-compact, connectedness, intersection theorem.
* This paper was partially supported by KOSEF in 1993 - 1994.

the theory of minimax, which contains a number of minimax theorems and the proofs of which require neither Hahn-Banach's theorem nor Brouwer's fixed point theorem.

In this paper, we shall prove a non-compact general minimax theorem in non-convex topological settings and next prove an intersection theorem which generalizes Horvath's result.

2. Preliminaries

Throughout this paper, we assume all topological spaces are Hausdorff. Let A be a subset of a topological space X. We shall denote by 2^A the family of all subsets of A. If A is a subset of a vector space, we shall denote by $co\,A$ the convex hull of A. If A is a non-empty subset of a topological space X and $T : A \to 2^X$ is a correspondence, then $T^{-1} : X \to 2^A$ is a correspondence defined by $T^{-1}(y) = \{x \in A : y \in T(x)\}$ for each $y \in X$.

Let X be a non-empty set in a topological space, Y a non-empty convex set in a vector space and $f : X \times Y \to R$ be a real-valued function. Then f is said to be *upper semicontinuous on the line segment of* Y if for each $x \in X$ and $y_1, y_2 \in Y$, the function

$$t \to f(x, ty_1 + (1-t)y_2)$$

is an upper semicontinuous function of t on $[0, 1]$.

If f is an upper semicontinuous function on Y, then f is clearly upper semicontinuous on the line segment of Y. In fact, the mapping $t \to ty_1 + (1-t)y_2$ is a bicontinuous mapping of t for each fixed $y_1, y_2 \in Y$. And f is said to be *lower semicontinuous on the line segment of* Y if $-f$ is upper semicontinuous on the line segment of Y.

Let X be a non-empty convex set in a vector space and f be a real-valued function defined on X. We recall that f is called *quasi-concave* if for any real number t, the set $\{x \in X : f(x) > t\}$ is convex ; and f is called *quasi-convex* if $-f$ is quasi-concave. Let C be a non-empty subset of X. We may call f is *quasi-concave on* C if the set $\{x \in C : f(x) > t\}$ is convex; and f is called *quasi-convex on* C if $-f$ is quasi-concave on

C. Other definitions and terminologies in this paper can be found in [7, 10].

3. A generalization of Horvath's intersection theorem

We begin with the following new minimax theorem of Horvath :

LEMMA [7]. *Let X be a non-empty compact topological space and Y be a non-empty convex subset of a vector space. Let $f : X \times Y \to R$ be a real-valued function satisfying the following conditions:*

(1) *for each fixed $y \in Y$, $x \to f(x,y)$ is a lower semicontinuous function of x on X;*
(2) *for each fixed $x \in X$, $y \to f(x,y)$ is a quasi-concave function of y on Y;*
(3) *f is upper semicontinuous on the line segment of Y;*
(4) *for each non-empty finite set $F \subseteq Y$ and each $\lambda \in R$,*

the set $\cap_{y \in F} \{x \in X : f(x,y) \leq \lambda\}$ is either connected or empty.

Then we have

$$\inf_{x \in X} \sup_{y \in Y} f(x,y) \leq \sup_{y \in Y} \inf_{x \in X} f(x,y).$$

REMARK. The condition (4) of Lemma is a weakening introduced by Geraghty-Lin [3] of the condition first introduced by Terkelson [14].

First we shall prove the following non-compact minimax theorem :

THEOREM 1. *Let X be a non-empty subset of a topological space and Y be a non-empty convex subset of a vector space. Let $f : X \times Y \to R$ be a real-valued function satisfying the following conditions:*

(1) *for each fixed $y \in Y$, $x \to f(x,y)$ is a lower semicontinuous function of x on X;*
(2) *for each fixed $x \in X$, $y \to f(x,y)$ is a quasi-concave function of y on Y;*
(3) *f is upper semicontinuous on the line segment of Y;*

(4) *for each non-empty finite set $F \subset Y$, there exists a non-empty compact subset C_F of X such that for each $\lambda \in R$ and each finite subset G of $co(F)$,*

the set $\cap_{y \in G} \{x \in C_F : f(x,y) \le \lambda\}$ *is either connected or empty,*

and, for each $y \in co(F)$,

$$(*) \qquad \inf_{x \in X \setminus C_F} f(x,y) \ge \max\{\inf_{x \in C_F} f(x,y), \inf_{x \in X} \sup_{y \in Y} f(x,y)\}.$$

Then we have

$$\inf_{x \in X} \sup_{y \in Y} f(x,y) \le \sup_{y \in Y} \inf_{x \in X} f(x,y).$$

Proof. Let $\alpha = \inf_{x \in X} \sup_{y \in Y} f(x,y)$, $\beta = \sup_{y \in Y} \inf_{x \in X} f(x,y)$ and suppose the contrary, i.e. $\alpha > \beta$. Then there exists $\epsilon > 0$ such that $\beta + \epsilon < \alpha$. Now let $K(y) := \{x \in X : f(x,y) \le \beta + \epsilon\}$ for each $y \in Y$. Then by the definition of β, each $K(y)$ is non-empty, and by the assumption (1), each $K(y)$ is closed in X.

Note that by the assumption (*), we can obtain that each $K(y)$ is compact. In fact, for each $y \in Y$, there exists a non-empty compact set C_y such that $\alpha = \inf_{x \in X} \sup_{y \in Y} f(x,y) \le \inf_{x \in X \setminus C_y} f(x,y)$, so that $K(y) \subset C_y$ and hence $K(y)$ is compact.

To obtain a contradiction, we must show that $\cap_{y \in Y} K(y)$ is non-empty. Therefore it suffices to show that the collection $\{K(y) : y \in Y\}$ has the finite intersection property. Let $F = \{y_1, \cdots, y_n\}$ be any finite subset of Y and C_F be the non-empty compact subset of X satisfying the condition (4). Now let

$$\bar{K}(y) := K(y) \cap (\cap_{i=1}^n \{x \in C_F : f(x,y_i) \le \beta + \epsilon\}) \quad \text{for each} \quad y \in co(F).$$

Then each $\bar{K}(y)$ is a compact subset of C_F. We first show that $\cap_{i=1}^n \bar{K}(y_i) \ne \emptyset$. Now we consider the restriction $f|_{C_F \times co(F)}$ of f on $C_F \times co(F) \subset X \times Y$; then the whole assumptions of Lemma are clearly satisfied, so that by Lemma, we have

$$\inf_{x \in C_F} \sup_{y \in co(F)} f(x,y) \le \sup_{y \in co(F)} \inf_{x \in C_F} f(x,y).$$

We note that the condition (*) implies $\inf_{x\in C_F} f(x,y) = \inf_{x\in X} f(x,y)$. By the assumption (*) again, we have

$$\begin{aligned}\inf_{x\in C_F}\sup_{y\in co(F)} f(x,y) &\le \sup_{y\in co(F)}\inf_{x\in C_F} f(x,y)\\ &= \sup_{y\in co(F)}\inf_{x\in X} f(x,y)\\ &\le \sup_{y\in Y}\inf_{x\in X} f(x,y)\\ &< \beta+\epsilon,\end{aligned}$$

so that there exists $x_o \in C_F$ such that $f(x_o, y) \le \beta+\epsilon$ for all $y \in co(F)$. Hence we have $x_o \in \cap_{y\in co(F)}K(y) \subset \cap_{i=1}^{n} K(y_i)$, so that the collection $\{K(y) : y \in Y\}$ of compact sets has the finite intersection property. Therefore we have $\cap_{y\in Y}K(y) \ne \emptyset$, so that for any $\bar{x} \in \cap_{y\in Y}K(y)$, we have $f(\bar{x}, y) \le \beta + \epsilon$ for all $y \in Y$. Hence $\inf_{x\in X}\sup_{y\in Y} f(x,y) \le \beta + \epsilon < \alpha$, which contradicts to the definition of α. This completes the proof.

REMARK. In Theorem 1, if for every finite subset F of Y, $C_F = X$ is compact, then the assumption (*) is automatically satisfied, and in this case, the assumption (4) is stronger than the corresponding assumption (4) of Lemma.

As in [7], we can obtain the following

COROLLARY 1. *Let X be a non-empty convex subset of a topological vector space and Y be a non-empty convex subset of a vector space. Let $f : X \times Y \to R$ be a real-valued function satisfying the following conditions:*

(1) *for each fixed $y \in Y$, $x \to f(x,y)$ is a lower semicontinuous function of x on X;*
(2) *for each fixed $x \in X$, $y \to f(x,y)$ is a quasi-concave function of y on Y;*
(3) *f is upper semicontinuous on the line segment of Y;*
(4) *for each non-empty finite set $F \subseteq Y$, there exists a non-empty compact subset C_F of X such that for each $\lambda \in R$ and each $y \in co(F)$,*

$$x \to f(x,y) \quad \text{is a quasi-convex function of } x \text{ on } C_F,$$

and

$$\inf_{x\in X\setminus C_F} f(x,y) \geq \max\{\inf_{x\in C_F} f(x,y), \inf_{x\in X}\sup_{y\in Y} f(x,y)\}.$$

Then we have

$$\inf_{x\in X}\sup_{y\in Y} f(x,y) \leq \sup_{y\in Y}\inf_{x\in X} f(x,y).$$

Proof. Since the function $x \to f(x,y)$ is quasi-convex on C_F for each $y \in co(F)$, then the set

$\cap_{y\in co(F)}\{x \in C_F : f(x,y) \leq \lambda\}$ is either convex or empty;

so that the assumption (4) of Theorem 1 is satisfied. Therefore we obtain the conclusion.

We can obtain the following generalization of von Neumann-Sion minimax theorem as a corollary:

COROLLARY 2. *Let X be a non-empty compact convex subset of a topological vector space and Y be a non-empty convex subset of a vector space. Let $f : X \times Y \to R$ be a real-valued function satisfying the following conditions:*

(1) *for each fixed $y \in Y$, $x \to f(x,y)$ is a lower semicontinuous and quasi-convex function of x on X;*
(2) *for each fixed $x \in X$, $y \to f(x,y)$ is a quasi-concave and upper semicontinuous on the line segment function of y on Y.*

Then we have

$$\inf_{x\in X}\sup_{y\in Y} f(x,y) \leq \sup_{y\in Y}\inf_{x\in X} f(x,y).$$

Let Y be a convex subset of a vector space and for each $y_1, y_2 \in Y$, $[y_1, y_2]$ denotes the line segment $\{ty_1+(1-t)y_2 : t \in [0,1]\}$, equipped with the Euclidean topology. Let A be a non-empty subset of a topological space X, then A is called *compactly closed in* X if $A\cap C$ is closed for every compact subset C of X.

Next we shall prove a generalization of Horvath's intersection theorem[7] in non-compact non-convex sets.

THEOREM 2. *Let X be a non-empty subset of a topological space, Y a non-empty convex subset of a vector space and let $T : Y \to 2^X$ be a correspondence satisfying the following:*

(1) *for each $y \in Y$, $T(y)$ is compactly closed in X;*
(2) *for each $x \in X$, $Y \setminus T^{-1}(x)$ is convex;*
(3) *for each $x \in X$ and $y_1, y_2 \in Y$, $T^{-1}(x) \cap [y_1, y_2]$ is open in $[y_1, y_2]$;*
(4) *for each $y \in Y$, there exists a non-empty compact subset C_y of X with $T(y) \cap C_y \neq \emptyset$ such that for each finite subset F of Y,*
(4i) $\cap_{y \in F} (T(y) \cap C_y)$ is either connected or empty,
(4ii) if $\cap_{y \in F} (T(y) \cap C_y) \neq \emptyset$, then $\cap_{y \in F} (T(y) \cap C_y) \cap C_z \neq \emptyset$ for each $z \in Y \setminus F$.

Then we have $\quad \cap_{y \in Y} T(y) \neq \emptyset$.

Proof. By the assumptions (1) and (4), for each $y \in Y$, $T(y) \cap C_y$ is non-empty compact, and we simply denote $T(y) \cap C_y$ by A_y. It suffices to show that the collection $\{A_y : y \in Y\}$ of compact sets has the finite intersection property. We now use the induction argument on the cardinality $|F|$ of any finite subset F of Y. When $|F| = 1$, we have done. Next, in case $|F| = 2$; suppose that $A_{y_1} \cap A_{y_2} = \emptyset$ for some $y_1, y_2 \in Y$. Define

$$\tilde{A}_t := T(ty_1 + (1-t)y_2) \quad \text{for each } t \in [0,1],$$

and let

$$U := \{t \in [0,1] : \tilde{A}_t \cap C_{y_2} \subset A_{y_2}\},$$

$$V := \{t \in [0,1] : \tilde{A}_t \cap C_{y_1} \subset A_{y_1}\}.$$

By the assumption that $A_{y_1} \cap A_{y_2} = \emptyset$, we have $U \cap V = \emptyset$. Next, we have $U \cup V = [0,1]$. In fact, suppose there exists $\tilde{t} \in (0,1)$ such that $\tilde{t} \in (U \cup V)^c = U^c \cap V^c$. Then there exists $x \in T(\tilde{t}y_1 + (1-\tilde{t})y_2)$ such that $x \notin T(y_1)$ and $x \notin T(y_2)$. Therefore $y_1, y_2 \in Y \setminus T^{-1}(x)$, and by the assumption (2), we have $\tilde{t}y_1 + (1-\tilde{t})y_2 \in Y \setminus T^{-1}(x)$, so that $x \notin T(\tilde{t}y_1 + (1-\tilde{t})y_2)$, which is a contradiction.

Now we show that $U = \{t \in [0,1] : (T(ty_1 + (1-t)y_2) \cap C_{y_2}) \subset (T_{y_2} \cap C_{y_2})\}$ is closed in $[0,1]$. Let (t_n) be a sequence in U, which converges to $t_o \in [0,1]$. Then for each $n \in N$, $(T(t_n y_1 + (1-t_n)y_2) \cap C_{y_2}) \subset (T(y_2) \cap C_{y_2})$, so that $(T(t_n y_1 + (1-t_n)y_2) \cap C_{y_2}) \cap (T(y_1) \cap C_{y_1}) = \emptyset$. By the assumption (4ii), for each $x \in (T(y_1) \cap C_{y_1}) \cap C_{y_2}$, $x \notin T(t_n y_1 + (1-t_n)y_2)$, which implies $t_n y_1 + (1-t_n)y_2 \notin T^{-1}(x)$. Therefore we have $t_n y_1 + (1-t_n)y_2 \notin \cup_{x \in T(y_1)\cap C_{y_1} \cap C_{y_2}} T^{-1}(x)$. By the assumption (3), $\cup_{x \in T(y_1)\cap C_{y_1} \cap C_{y_2}} T^{-1}(x) \cap [y_1, y_2]$ is open in $[y_1, y_2]$, and hence $t_o y_1 + (1-t_o)y_2 \notin \cup_{x \in T(y_1)\cap C_{y_1} \cap C_{y_2}} T^{-1}(x)$.

This implies that $t_o y_1 + (1-t_o)y_2 \notin T^{-1}(x)$ for all $x \in T(y_1) \cap C_{y_1} \cap C_{y_2}$, and hence $x \notin T(t_o y_1 + (1-t_o)y_2)$ for all $x \in T(y_1) \cap C_{y_1} \cap C_{y_2}$. Therefore we have

$$\begin{aligned} & T(t_o y_1 + (1-t_o)y_2) \cap (T(y_1) \cap C_{y_1} \cap C_{y_2}) \\ &= (T(t_o y_1 + (1-t_o)y_2) \cap C_{y_2}) \cap (T(y_1) \cap C_{y_1}) \\ &= \emptyset. \end{aligned}$$

Hence $(T(t_o y_1 + (1-t_o)y_2) \cap C_{y_2}) \subset (T(y_2) \cap C_{y_2})$, so that $t_o \in U$. Therefore U is closed. Similarly, we can show that V is closed in $[0,1]$. Since $0 \in U$ and $1 \in V$, $\{U, V\}$ is a non-empty closed separation of $[0,1]$, which is a contradiction. Therefore we have $A_{y_1} \cap A_{y_2} \neq \emptyset$ for each $y_1, y_2 \in Y$.

Suppose that any subcollection of n elements of $\{A_y : y \in Y\}$ has non-empty intersection. Now we must show that any collection of $n+1$ elements of $\{A_y : y \in Y\}$ has also non-empty intersection. For any $y_o, y_1, \cdots, y_n \in Y$, we suppose that $\cap_{i=1}^n A_{y_i} \neq \emptyset$ and $\cap_{\substack{i=0 \\ i\neq 1}}^n A_{y_i} \neq \emptyset$. Then it suffices to show that $\cap_{i=0}^n A_{y_i} \neq \emptyset$.

We now define

$$\bar{A}_t := T(ty_0 + (1-t)y_1) \cap (\cap_{i=2}^n A_{y_i}) \quad \text{for each } t \in [0,1].$$

And we let

$$\bar{U} := \{t \in [0,1] : (\bar{A}_t \cap C_{y_1}) \cap (\cap_{i=2}^n A_{y_i}) \subset A_{y_1} \cap (\cap_{i=2}^n A_{y_i})\},$$
$$\bar{V} := \{t \in [0,1] : (\bar{A}_t \cap C_{y_o}) \cap (\cap_{i=2}^n A_{y_i}) \subset A_{y_o} \cap (\cap_{i=2}^n A_{y_i})\}.$$

Now suppose that $\cap_{i=0}^n A_{y_i} = \emptyset$; then by repeating the preceeding argument, we can obtain the following:

(a) $\bar{U} \cup \bar{V} = [0,1]$,

(b) $\bar{U}$ and $\bar{V}$ are non-empty closed sets, and

(c) $\bar{U} \cap \bar{V} = \emptyset$.

Therefore we have a contradiction. Hence we can obtain $\cap_{i=0}^{n} A_{y_i} = \cap_{i=0}^{n}(T_{y_i} \cap C_{y_i}) \neq \emptyset$. By the induction, the collection $\{A_y : y \in Y\}$ of compact sets has the finite intersection property, and hence $\emptyset \neq \cap_{y\in Y} A_y = \cap_{y\in Y}(T(y) \cap C_y) \subset \cap_{y\in Y} T(y)$, which completes the proof.

REMARK. We can replace the assumption (3) by the following without affecting the conclusion:

(3′) for each $x \in X$ and $y_1, y_2 \in Y$, $T^{-1}(x) \cap [y_1, y_2]$ is closed in $[y_1, y_2]$.

When X is compact, we let $C_y = X$ for each $y \in Y$; then the condition (4 ii) is automatically satisfied. In this case, we can obtain Theorem 3 in [7] as a corollary:

COROLLARY 3. *Let X be a non-empty compact subset of a topological space, Y a non-empty convex subset of a vector space and let $T : Y \to 2^X$ be a correspondence satisfying the following:*

(1) *for each $y \in Y$, $T(y)$ is non-empty closed in X;*
(2) *for each $x \in X$, $Y \setminus T^{-1}(x)$ is convex;*
(3) *for each $x \in X$ and $y_1, y_2 \in Y$, $T^{-1}(x) \cap [y_1, y_2]$ is open in $[y_1, y_2]$;*
(4) *for every finite subset F of Y,*
$\cap_{y\in F} T(y)$ *is either connected or empty.*

Then we have $\cap_{y\in Y} T(y) \neq \emptyset$.

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