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ON THE CHARACTERISTIC RING–MODULES

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From now on, we assume that a ring R has an identity 1. We have the following Lemma from Park[2].

LEMMA 1. *Let M be a right R-module. Then we have the followings :*

(1) *μ_M and μ_m are right congruences on R where $(a,b) \in \mu_M \Longleftrightarrow T_a = T_b$, $(a,b) \in \mu_m \Longleftrightarrow T_a(m) = T_b(m)$ and T_a, T_b are right translations.*
(2) *R/μ_M and R/μ_m are rings and right R-modules.*
(3) *T_R is a ring and right R-module where $T_R = \{T_a : a \in R\}$.*
(4) *L_M is a right R-module but not a ring where R is commutative, $L_M = \{L_m : m \in M\}$ and $L_m : R \longrightarrow M$ is a map defined by $L_m(r) = mr$.*

NOTATION. For simplicity, we shall denote R/μ_M and R/μ_m as $\chi(M)$ and $\chi(m)$ respectively.

DEFINITION 1. (1) $\chi(M)$ is called the characteristic ring-module with respect to a module M (or M-characteristic ring-module).

(2) M is a strong cyclic R-module if every element of M is a generator.

LEMMA 2. *Let M be a right R-module. Then*

(1) *$\chi(M) \cong T_R(R\text{-iso.})$;*
(2) *$\chi(M) \approx T_R(\text{ring-iso.})$ if R is commutative.*

We have introduced an M-reduced ring in Park[2]. We recall that R ia an M-reduced ring (or a reduced ring with respect to M) if for any $a, b \in R, T_a = T_b$, then $a = b$.

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PROPOSITION 3. *The following statements are equivalent :*

(1) *$\chi(M)$ is strong cyclic module ;*
(2) *for any $a, b \in R, (ar, b) \in \mu_M$ for some $r \in R$;*
(3) *(any μ_M-class) $\cap$ (any right ideal of R) $\neq \emptyset$ if R is an M-reduced ring.*

Proof. (1)$\Longleftrightarrow$ (2) : For any $a, b \in R$, $\chi(M)$ is strong cyclic $\Longleftrightarrow$ $[a]r = [b]$ for some $r \in R \Longleftrightarrow [ar] = [b] \Longleftrightarrow (ar, b) \in \mu_M$. (1)$\Longrightarrow$(3) : For any μ_M-class $[a] \in \chi(M)$ and right ideal $I \subset R$, let $[x]$ be an μ_M-class such that $I \cap [x] \neq \emptyset$. Then $[a] = [x]r$ for some $r \in R$. Hence $I \cap [a] \neq \emptyset$. (3) $\Longrightarrow$(1) : We note that $[a]R$ is a right ideal of R for $[a] \in \chi(M)$. From the assumption $[a]R \cap [b] \neq \emptyset$ for $[a], [b] \in \chi(M)$. Hence $[a]r = [b]$ for some $r \in R$.

We introduce new definitions.

DEFINITION 2. Let M be a right R-module.

(1) M is *invertible* if for any $a \in R$, $\exists b \in R$ such that $T_{ab} = T_1$ or $T_{ba} = T_1$ where '1' is the identity of R.
(2) M is *regular* if for any $a \in R, \exists b \in R$ such that $T_a = T_{aba}$.
(3) M is *element-independent* if $\mu_m = \mu_n$ for all $m, n \in M$.

PROPOSITION 4. *Let M be a right R-module. Then the following statements hold :*

(1) *If M is invertible, then M is regular ;*
(2) *M is regular $\Longleftrightarrow$ $\chi(M)$ is regular semigroup ;*
(3) *M is invertible $\Longleftrightarrow$ M is regular if M is element-independent;*
(4) *M is invertible $\Longleftrightarrow$ $\chi(M)$ is a group.*

Proof. (1) and (2) are clear. For (3), ($\Leftarrow$) : From the regularity of M, for any $a \in R$ $\exists b \in R$ such that $T_a = T_{aba}$. Now, let $q = pa$ for any $p \in M$. Then $q(ba) = pa(ba) = p(aba) = pa = q = q1$. So, we have $T_{ba}(q) = T_1(q)$. Since M is element- independent, we have $m(ba) = m1 = m$ for all $m \in M$. ($\Rightarrow$) : It holds from (1). For (4) ($\Rightarrow$) : It holds from (2) and the fact that if the regular semigroup has only one idempotent, that is a group. ($\Leftarrow$) : It ia trivial.

We have defined the perfect R-module in Park[2]. The following proposition holds from Park[2].

PROPOSITION 5. *Let M be a right R-module. then $\chi(M)$ and $Aut_R(M)$ are group-isomorphic if M is perfect.*

LEMMA 6. *Let N be a right R-module. Let α be a right congruence on R. If $f : R/\alpha \longrightarrow N$ is an R- epimorphism, then there is a right congruence β on R such that $N \cong R/\beta(R\text{-iso.})$ and $\alpha \subset \beta$.*

PROPOSITION 7. *Let M be a right R-module. $\chi(M)$ has a nontrivial R-homomorphism $\Longleftrightarrow$ there is a right congruence α on R such that $\mu_M < \alpha < 1$.*

Proof. $(\Rightarrow)$: Let $f : \chi(M) \longrightarrow N$ be a nontrivial R-homomorphism where N is a right R-module. We define a relation α on R by $(a, b) \in \alpha \Longleftrightarrow f([a]) = f([b])$ where [ ] means the μ_M-class. Claim : α is the desired one. We shall show this. From Lemma 6 we know that α is a right congruence on R with $\mu_M < \alpha$. Since f is nontrivial, there are [a], [b], [c] and [d] $\in \chi(M)$ such that

$$f([a]) = f([b]) \cdots\cdots\cdots\cdots\cdots\cdots (*)$$

and

$$f([c]) \neq f([d]) \cdots\cdots\cdots\cdots\cdots\cdots (**)$$

where $[a] \neq [b]$. Now, from (*) we have $(a, b) \in \alpha$ but $(a, b) \notin \mu_M$. Hence $\mu_M < \alpha$. From (**) we have $(c, d) \notin \alpha$ but $(c, d) \in 1$. Hence $\alpha < 1$. $(\Leftarrow)$: let $f : \chi(M) \longrightarrow R/\alpha$ be a map defined by $f([a]) = [a]_\alpha$. Then it is easy to check that f is an epimorphism. From $\mu_M < \alpha$ there is $(a, b) \in R \times R$ such that $(a, b) \in \alpha$ but $(a, b) \notin \mu_M$. Hence $f([a]) = f([b])$ does not implies $[a] = [b]$. So, f is not 1 - 1. Now, since $\alpha \neq 1$, R/α has at least two elements. This means $|R/\alpha| \neq 1$.

Let M be a right R-module. we define the equivalence relations μ_R and μ_a on M by $(m, q) \in \mu_R \Longleftrightarrow L_m = L_q$ and $(m, q) \in \mu_a \Longleftrightarrow L_m(a) = L_q(a)$ for $a \in R$. Then we note that μ_R and μ_a are congruence relations on M if R is commutative. Also, we note that M/μ_R is a right R-module.

DEFINITION 3. M/μ_R is called a *characteristic right R- module* and we denote $M/\mu_R = \chi(R)$.

PROPOSITION 8. *Let M be a right R-module. Then $\chi(R)$ has a nontrivial R-homomorphism $\Longleftrightarrow$ there is a congruence α on M such that $\mu_R < \alpha < 1$.*

Proof. $(\Rightarrow)$: Let $f : \chi(R) \longrightarrow N$ be a nontrivial R-homomorphism where N is a right R-module. We define a relation α on M by $(x, y) \in \alpha \Longleftrightarrow f([x]) = f([y])$ where [ ] means the μ_R-class. Then α is the desired one. $(\Leftarrow)$: We can prove this part in the same way as proposition 5.

DEFINITION 4. Let M be a right R-module. M is *UI-primitive* if 0 and 1 are the only right congruences on M where U means "universal" and I means "identity".

PROPOSITION 9. *Let M be a right R-module where R is commutative ring. Then the following statements hold :*

(1) $\chi(M) \cong L_M$ *(R-iso.) ;*

(2) $M \cong L_M$ *if M is UI-primitive and $|L_M(a)| \neq 1$ for some $a \in R$;*

(3) $L_M = Hom(R, M)$.

Proof. For (1) we define $h : \chi(R) \longrightarrow L_M$ by $h([m]) = L_m$ for $m \in M$. For (2) we know that μ_a is a right congruence on M. From the UI-primitiveness of M we have $\mu_a = 0$ or $\mu_a = 1$. Suppose $\mu_a = 1$. Then $L_M(a) = Ma = \{ma : m \in M\}$ and $|L_M(a)| = 1$. This is a contradiction. From $\mu_R = \bigcap_{a \in R} \mu_a$ it holds. For (3) choose any $g \in$ Hom(R, M). Then $L_{g(1)}(r) = g(1)r = g(1r) = g(r)$ for any $r \in R$. Hence $g \in L_M$. The converse is clear.

PROPOSITION 10. *Let M be a cyclic R-module. Then M is strongly cyclic $\Longleftrightarrow$ $\chi(m)$ is a strongly cyclic R-module for each $m \in M$.*

Proof. $(\Rightarrow)$: Let $M = zR$ for some $z \in M$. We define $f : \chi(m) \longrightarrow M = zR$ by $f([a]) = za$. Then it is easy to check that f is an R-isomorphism. $(\Leftarrow)$: It is trivial.

DEFINITION 5. Let M be a right R-module. (1) M is ***of group type*** if M is element-independent and $\chi(M)$ is a group. (2) M is ***quasi-perfect*** if M is strongly cyclic and of group type.

If M is perfect, we have $T_R = Aut_R(M)$. From Corollary 3.2 and Proposition 6 of Park[2] we have the following Proposition.

PROPOSITION 11. *Let M be a right R-module. If M is perfect, then M is quai-perfect.*

DEFINITION 6. Let M be a right R-module and let $H \in p(M) - \emptyset$ where $p(M)$ is the power set of M and $a \in R$. Then we define the sets $\chi(M)_H$ and $\chi(M)_{Ha}$ as follows :

$$\chi(M)_H = \{ \ [a] \in \chi(M) : Ha = H \ \}$$

and

$$\chi(M)_{Ha} = \{ \ [r] \in \chi(M) : (Ha)r = Ha \ \}.$$

PROPOSITION 12. *Let M be a right R-module. Then the following statements hold :*

(1) *$\chi(M)_H$ is a subgroup of $\chi(M)$ if M is invertible ;*
(2) *$\chi(M)_H$ is a normal subgroup of $\chi(M)$ if M is invertible and R commutative.*

Proof. For (1) we shall show $[a][b]^{-1} \in \chi(M)_H$ for any $[a],[b] \in \chi(M)_H$. Now, from $[a],[b] \in \chi(M)_H$ we have $[a],[b] \in \chi(M)_H$ with $Ha = H$ and $Hb = H$. So, there is $[z] \in \chi(M)$ such that $[b][z] = [1]$. Let $[b]^{-1} = [z]$. Then $[b]^{-1} \in \chi(M)_H$ and $[a][b]^{-1} \in \chi(M)_H$. For (2) let $[a]^{-1} = [b]$ for $[a] \in \chi(M)$. Choose any $[x] \in \chi(M)_H$. Then $[a][x][b] \in \chi(M)$ and $Haxb = H$. Hence it holds.

LEMMA 13. *Let M be a quasi-perfect right R-module and let $H \in p(M) - \emptyset$. Then for $a, b \in R$, $Ha = Hb \Longleftrightarrow \chi(M)_H[a] = \chi(M)_H[b]$.*

Proof. $(\Rightarrow)$: We shall show $\chi(M)_H[a] \subset \chi(M)_H[b]$. To do this, it is enough to show $[a] = [r][b]$ for some $[r] \in \chi(M)_H$. Now, for any $h \in H$ $ha = qb$ for some $q \in H$. Since M is strongly cyclic, $q = hr$ for $r \in R$. From the element- independence of M and $ha = hrb$, $[a] = [r][b]$ holds. Similarly, we can prove the converse. $(\Leftarrow)$: $T_{ta} = T_{ub}$ for some $[t],[u] \in \chi(M)_H$. Hence it holds.

PROPOSITION 14. *Let M be a quasi-perfect right R-module. and let $H \in p(M) - \emptyset$. Then $\chi(M)_{Ha} = [\chi(M)_H]^{[a]}$ where $[\chi(M)_H]^{[a]} = [a]^{-1}\chi(M)_H[a]$.*

Proof. Let $[t] \in \chi(M)_{Ha}$. Then $Hat = Ha$ and from Lemma 13 $\chi(M)_H[at] = \chi(M)_H[a]$. Hence $[a][t] = [r][a]$ for some $[r] \in \chi(M)_H$. From this $[t] \in [\chi(M)_H]^{[a]}$. Conversely, let $[x] = [a]^{-1}[r][a]$ for some $r \in \chi(M)_H$. Then $Hr = H$ and $Hax = Ha$. Hence $[x] \in \chi(M)_{Ha}$.

DEFINITION 7. Let M be a right R-module and let $H \in p(M) - \emptyset$. Then we define the set $\chi(M)_{HR}$ as follows :

$$\chi(M)_{HR} = \{\ [r] \in \chi(M) : (Ha)r = Ha \ \ for \ \ all \ \ a \in R\}.$$

NOTE.

$$\chi(M)_{HR} = \bigcap_{a \in R} \chi(M)_{Ha}$$

PROPOSITION 15. *Let M be an invertible right R-module and let $H \in p(M) - \emptyset$. Then the following statements hold :*

(1) *$\chi(M)_{HR}$ is a normal subgroup of $\chi(M)$;*
(2) *if $N(\neq \emptyset), N^{[a]} \subset \chi(M)_H$ for any $[a] \in \chi(M)$, then $N, < N > \subset \chi(M)_{HR}$ where $< N >$ is the subgroup of $\chi(M)$ generated by N and $N^{[a]} = [a]N[a]^{-1}$*

Proof. For (1) choose $[r] \in \chi(M)_{HR}$ and $[a] \in \chi(M)$. Let $[a]^{-1} = [z]$. Then $[a][r][z] \in \chi(M), [a][z] = [1]$ and for any $t \in R, (Ht)arz = (Hta)rz = (Hta)z = (Ht)az = Ht$. Hence it holds. For (2) $N \subset \chi(M)_{HR}$: Choose any $[t] \in N$ and $a \in R$. Let $[a]^{-1} = [z]$. Then $[a][t][z] \in \chi(M)_H$. From this we have $Hatz = H$. Hence $Ha = (Ha)tza = (Ha)t$. This implies $[t] \in \chi(M)_{HR}$. Next, $< N > \subset \chi(M)_{HR}$: (i) for any $[r], [t] \in N$, we shall show $[r][t] \in \chi(M)_{HR}$. For any $a \in R$, let $[a]^{-1} = [z]$. Then $Hartz = H$ and $Ha = (Hartz)a = Hartza = (Ha)rt$. Hence it holds. (ii) for any $[r] \in N, [r]^{-1} \in \chi(M)_{HR}$. To show this, for any $a \in R$ let $[a]^{-1} = [u]$ and $[r]^{-1} = [z]$. Then $Haru = H$ and $Har = Ha$. Hence $(Ha)z = (Har)z = Harz = Ha$.

Let N be a normal subgroup of $\chi(M)$ with $N \subset \chi(M)_H$. Then for $[a] \in \chi(M)$, $N^{[a]} \subset N \subset \chi(M)_H$. Therefore we have the following Corollary.

COROLLARY 15.1. *Let M be an invertible right R-module and let $H \in p(M) - \emptyset$. If N is a normal subgroup of $\chi(M)$ with $N \subset \chi(M)_H$, then $N \subset \chi(M)_{HR}$.*

PROPOSITION 16. *Let M be an invertible right R-module and let $H \in p(M) - \emptyset$. Then $\chi(M)_{HR} = \chi(M)_H$ if R is commutative.*

PROPOSITION 17. *Let M be a quasi-perfect right R-module and let K be a subgroup of $\chi(M)$. then $K = \chi(M)_H$ for some $H \in p(M) - \emptyset$.*

Proof. Let $H = \{ mk : [k] \in K\}$. Let $[a] \in \chi(M)_H$. Then $Ha = H$ and $mka = mb$ for some $[k], [b] \in K$. Since M is element- independent, $[k][a] = [b]$. Hence $[a] = [k]^{-1}[b] \in K$. The converse is trivial.

COROLLARY 17.1. *Let M be q quasi-perfect right R-module and let N be a normal subgroup of $\chi(M)$. Then $N = \chi(M)_{HR}$ for some $H \in p(M) - \emptyset$.*

PROPOSITION 18. *Let M be a quasi-perfect right R-module and let H be a submodule of M. Then $\chi(M)$ and $\chi(M)/\chi(M)_H$ are group-isomorphic.*

Proof. Let $f : \chi(M) \longrightarrow \chi(M)/\chi(M)_H$ be a map given by $f(\{a\}) = \chi(M)_H[a]$ for $a \in R$. (i) f is well-defined : It follows from Lemma 13. (ii) f is 1 - 1 : Let $f(\{a\}) = f(\{b\})$. Then $[u][a] = [w][b]$ for some $[u], [w] \in \chi(M)_H$ and $Hu = H, Hw = H$. $[ua] = [wb] \Leftrightarrow T_{ua} = T_{wb}$ on $M \Rightarrow T_{ua}(H) = T_{wb}(H) \Leftrightarrow Hua = Hwb \Rightarrow Ha = Hb \Leftrightarrow T_a(H) = T_b(H) \Leftrightarrow \{a\} = \{b\}$. (iii) f is onto : It is clear. (iv) f is group-homomorphism : It is trivial.

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