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A CONJUGACY THEOREM IN PROFINITE GROUPS

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1.Introduction

Two subgroups U and V of a finite group G are called to be p-conjugate for a prime p if a Sylow p-subgroup of U is conjugate to a Sylow p-subgroup of V. This concept of p-conjugacy also makes sense for some infinite groups with a reasonable Sylow theory.

By $\mathbf{N}^r$ we mean the class of finite groups which have nilpotent length at most r. The $\mathbf{N}^r$-residual of a finite group G is defined by

$$\mathbf{N}^r(G) = \cap\{H \trianglelefteq G : G/H \in \mathbf{N}^r\}$$

In 1956, G. Higman showed that for any finite group G, if $\mathbf{N}^r(G)$ is abelian then it is complemented, and any two complements are conjugate[4].

In 1979, in their attempt to generalize this result, Loesy and Stonehewer proved the following theorem[5]:

THEOREM 1.1. *Let G be a finite sovable group. Let two subgroups U and V of G be p-conjugate for every prime p. Suppose that U and V have a nilpotent common normal supplement X(that is, $G = UX = VX$ and X is nilpotent and normal in G) and that one of the following conditions is satisfied :*

1. *X is abelian,*
2. *G/X is nilpotent, or*
3. *the Sylow p-subgroups of G have class at most 2 for every prime p.*

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Then U and V are conjugate

In [2], it was shown that Losey-Stonehewer theorm holds without solvability. It is desired to generalize the theorem for infinite groups. In fact we have some generalizations for specific classes of locally finite groups[7]. In this note, we are concerned with profinite groups.

2. Profinite group

An inverse limit of finite groups is called a profinite group. Since every finite group is a compact Hausdorff group, a profinite group is also a compact Hausdorff group. In fact, the following theorem characterizes the profinite groups[6].

THEOREM 2.1. *The following conditions are equivalent :*

(1) *G is a profinite group.*
(2) *G is a compact, Hausdorff group in which the family of open normal subgroups forms a fundamental system of neibourhood at 1.*
(3) *G is a compact, totally disconnected, Hausdorff group.*

The following lemma is useful for our purpose.

LEMMA 2.2. *If S is a subgroup of a profinite group G, then its closure $\bar{S}$ is given by*

$$\bar{S} = \cap\{SU \mid U \text{ is a normal subgroup of finite index in } G\}.$$

Proof. Every open subgroup of G is closed [3]. From Theorem 2.1, we know that

$$\bar{S} = \cap\{U \mid U \text{ is an open normal subgroup of } G \text{ and } S \subseteq U\}.$$

but

$$\begin{aligned} \bar{S} &\leq \cap\{SU \mid U \text{ is a normal subgroup of finite index in } G\} \\ &\leq \cap\{U \mid U \text{ is an open normal subgroup of } G \text{ and } S \subseteq U\}. \end{aligned}$$

This proves the lemma.

The normalizer of a closed group in profinite groups plays an important role in the proof of our Theorem.

LEMMA 2.3. *If G is a profinite group and let F be a closed subgroup of G. Then $N_G(F)$, the normalizer of F in G, is closed.*

Proof. Let U be a open normal subgroup of G and $N_{G/N}(FU/U) = F_U/U$. Then F_U is closed. We show

$$N_G(F) = \cap\{F_U : U \text{ is an open normal subgroup of G }\}.$$

Clearly

$$N_G(F) \subseteq \cap\{F_U \mid U \text{ is an open normal subgroup of } G\}.$$

If

$$x \in \cap\{F_U \mid U \text{ is an open normal subgroup of } G\},$$

then $F^xU = FU$ for all open normal subgroup U of G. By Lemma 2.2,

$$F^x = \bar{F}^x = \cap\{F^xU \mid U \text{ is an open normal subgroup of } G\} = \bar{F} = F.$$

Hence $x \in N_G(F)$.

A profinite group has a reasonable Sylow theory. For a brief explanation of it, we need the following :

DEFINITION 2.4. A supernatural number is a formal product $\prod_{p:\text{prime}} p^{n_p}, 0 \leqslant n_p \leqslant \infty$.

Notice that ∞ is allowed for exponents. Supernatural numbers can be multiplied in a natural way(i.e., by addition of exponents). Also it is easy to define the least common multiple(l.c.m) and the greatest common divisor(g.c.d) of supernatural numbers.

DEFINITION 2.5. Let G be a profinite group, H a closed subgroup. The index of H in G(denoted by $[G : H]$) is defined by :

$$[G : H] = \text{l.c.m.}\{[G : U] | U \text{ ranges over open subgroups of } G \text{ containing } H\}.$$

And the order of G (denoted by $|G|$) is defined to be $[G : 1]$.

We remark that

$$[G : H] = \text{l.c.m.}\{[G : HN] | N \text{ ranges over the open normal subgroups of } G\}.$$

If s is a supernatural number, s is called a p-power (or prime-to-p) if $n_q = 0$ for all $q \neq p$ (or $n_p = 0$). And we say that G is a p-group if $|G|$ is a p-power.

For a generalization of these concepts, the readers are referred to [1].

DEFINITION 2.6. Let S be a closed subgroup of a profinite froup G. S is called a Sylow p-subgroup of G if $[G : S]$ is prime-to-p.

For the completeness of our discussion we include the following Sylow theorem for profinite groups. The proof can be found in [6].

THEOREM2.7. *Let G be a profinite group, and let p be a prime number. Then*

(1) *G possesses Sylow p-subgroups:*
(2) *If T is any p-subgroup of G then T is contained in a Sylow p-subgroup of G.*
(3) *Any two Sylow p-subgroups of G are conjugate.*

3. Theorem

Now we state and prove our Theorem.

THEOREM. *Let G be a profinite group. Let two subgroups U and V of G be closed and p-conjugate for every prime p. Suppose that U and V have a closed, nilpotent common normal supplement X and that one of the following conditions is satisfied :*

(1) *X is abelian;*
(2) *G/X is nilpotent;*
(3) *the Sylow p - subgroups of G have class at most 2 for every prime p.*

Then U and V are conjugate

Proof. Let $\{N_\alpha\}$ be the family of all normal subgroups of finite index in G. For each α, consider the cannonical homomorphism.

$$\phi_\alpha : G \longrightarrow G/N_\alpha$$

Since G/N_α is finite, we can apply the theorem of [2] in

$$\begin{aligned} G/N_\alpha &= UN_\alpha/N_\alpha \cdot XN_\alpha/N_\alpha \\ &= VN_\alpha/N_\alpha \cdot XN_\alpha/N_\alpha \end{aligned}$$

to obtain an element $x_\alpha N_\alpha \in G/N_\alpha$ such that

$$(UN_\alpha/N_\alpha)^{x_\alpha N_\alpha} = VN_\alpha/N_\alpha$$

This means that $(UN_\alpha)^{x_\alpha} = VN_\alpha$. If we let $F_\alpha = \{x \in G | (UN_\alpha)^x = VN_\alpha\}$, then $F_\alpha = x_\alpha N_G(U)$. From the above argument and Lemma 2.3, F_α is non-empty and closed. Now it is clear that $F_\alpha \subseteq F_\beta$ if $N_\alpha \subseteq N_\beta$. So $\{F_{N_\alpha}\}$ has the finite intersection property. The compactness of G implies that $\cap_\alpha F_\alpha \neq \phi$. If we take $x \in F_\alpha$, then

$$\begin{aligned} V = \bar{V} &= \cap\{VN_\alpha\} = \cap\{(UN_\alpha)^x\} \\ &= (\cap\{UN_\alpha\})^x = \bar{U}^x = U^x. \end{aligned}$$

Therefore, U and V are conjugate.

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