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AREA INTEGRALS WITH A MEASURE ON GROUPS OF HOMOGENEOUS TYPE

CHOON-SERK SUH

1. Introduction

We define a group of homogeneous type G which is a more general setting than R^n. This group G forms a natural habitat for extensions of many of the objects studied in Euclidean harmonic analysis.

In 1985, R. R. Coifman, Y. Meyer and E. M. Stein ([1], [2]) introduced the tent spaces on the upper half-space R_+^{n+1} in the Euclidean space. In this paper, we will develop the theory of the tent spaces on $G \times (0, \infty)$, which is an analogue of the upper half-space R_+^{n+1}. Also, we are concerned with a relation between area integrals A and W_p on $G \times (0, \infty)$. If now f is a function on $G \times (0, \infty)$, then we define

$$A(f)(x) = \left\{ \int_{\Gamma(x)} |f(y,t)|^2 \frac{d\mu(y)dt}{t^{n+1}} \right\}^{1/2},$$

and for $0 < p \leq 1$

$$W_p(f)(x) = \left\{ \int_{\Gamma(x)} |f(y,t)|^2 \frac{t^{2(n-m)/p}}{t^{n+1}} d\mu(y)dt \right\}^{1/2},$$

where $\Gamma(x)$ is the cone with vertex at $x \in G$.

Our main result is that there exists a constant C_p so that if ν is a positive measure on G satisfying $\nu(B(x,\rho)) \leq C\rho^m$, then we have

$$||W_p(f)||_{L^p(d\nu)} \leq C_p ||A(f)||_{L^p(d\mu)},$$

where $0 < p \leq 1$ and $d\mu$ denotes a positive measure on G.

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2. Preliminaries

Let G be a topological group endowed with a positive measure μ on G. Assume that d is a pseudo-distance on G, i.e., a non-negative function defined on $G \times G$ satisfying

(i) $d(x,x) = 0$; $d(x,y) > 0$ if $x \neq y$,
(ii) $d(x,y) = d(y,x)$, and
(iii) there is a constant K such that $d(x,z) \leq K[d(x,y) + d(y,z)]$, $\forall x, y, z, \in X$.

Assume further that

(G1) the balls $B(x,\rho) = \{y \in G : d(x,y) < \rho\}$, $\rho > 0$, form a basis of open neighborhoods at $x \in G$, and

(G2) there is a constant C such that

$$0 < \mu(B(x,2\rho)) \leq C\mu(B(x,\rho)) < \infty,$$

for all $x \in X, \rho > 0$.
Finally we assume that μ is left-invariant:

(G3) $\mu(xE) = \mu(E)$ for each $x \in G$ and any measurable $E \subset G$, and

(G4) $\mu(E^{-1}) = \mu(E)$,

and that d is left-invariant:

(G5) $xB(y,\rho) = B(xy,\rho)$ for all $x, y \in G$ and $\rho > 0$.

Then we call (G,d,μ) a *group of homogeneous type.*

Let (G,d,μ) be a group of homogeneous type. Then for $\rho > 0$, an automorphism δ_ρ of G is called a *dilation* on G if

$$\mu(\delta_\rho(E)) = \rho^n \mu(E)$$

for some fixed positive integer n and any measurable $E \subset G$, and in particular,

$$\mu(\delta_\rho(B(x,1))) = \mu(B(e,\rho)) = C_n \rho^n,$$

where C_n denotes the volume of the unit ball $B(e,1)$. Actually, most of the time we shall write ρx instead of $\delta_\rho x$ for $\rho > 0$ and $x \in G$.

It is known that d is left-invariant if and only if

$$d(x,y) = |x^{-1}y|,$$

where $|\cdot|$ is a nonnegative function on G satisfying

(i) $|x| = 0$ if and only if $x = e$,
(ii) $|xy| \leq K(|x| + |y|)$, where K is some fixed constant, and
(iii) $|x^{-1}| = |x|$.

For details see [7].

For $x, y \in G$, and $\rho > 0$, the set

$$B(x,\rho) = \{y \in G : |x^{-1}y| < \rho\}$$

is called the *ball* centered at $x \in G$ with radius ρ.

Now consider the space $G \times (0,\infty)$, which is called the *upper half-space* over G. For each $x \in G$, and $\alpha > 0$, the set

$$\Gamma_\alpha(x) = \{(y,t) \in G \times (0,\infty) : |x^{-1}y| < \alpha t\}$$

is called the *cone* of aperture α whose vertex is $x \in G$. For simplicity, we put $\Gamma(x) = \Gamma_1(x)$.

For any closed subset $F \subset G$, and $\alpha > 0$, let

$$\mathcal{R}^{(\alpha)}(F) = \cup\{\Gamma_\alpha(x) : x \in F\}.$$

For simplicity, we put $\mathcal{R}(F) = \mathcal{R}^{(1)}(F)$. Let O be an open subset of G which is the complement of F, $O = F^c$. Then the *tent* over O, denoted by $T(O)$, is given as $T(O) = \mathcal{R}(F)^c$.

For a function f defined on $G \times (0,\infty)$, we define a functional $A(f)$, for $x \in G$, by

$$A(f)(x) = \left\{ \int_{\Gamma(x)} |f(y,t)|^2 \frac{d\mu(y)dt}{t^{n+1}} \right\}^{1/2}.$$

Then the *tent space* $T_2^p(G \times (0,\infty))$ is defined as the space of functions f on $G \times (0,\infty)$, so that $A(f) \in L^p(d\mu)$, when $0 < p < \infty$. Define

$$||f||_{T_2^p} = ||A(f)||_{L^p(d\mu)}.$$

For $0 < p \leq 1$, a function $a(x,t)$, supported in $T(B)$ for some ball B in G, is said to be a T_2^p-*atom* if

$$\int_{T(B)} |a(x,t)|^2 \frac{d\mu(x)dt}{t} \leq [\mu(B)]^{1-2/p}.$$

LEMMA 1 ([9]). *There exists a constant C so that if $f \in T_2^p(G \times (0,\infty))$ for $0 < p \leq 1$, then there exist a sequence $\{a_j\}$ of T_2^p-atoms, and a sequence $\{\lambda_j\}$ of positive numbers such that*

$$|f(x,t)| \leq \sum_{j=1}^{\infty} \lambda_j a_j(x,t),$$

and

$$\sum_{j=1}^{\infty} \lambda_j^p \leq C\|A(f)\|_{L^p(d\mu)}^p.$$

We are now going to generalize $A(f)$. This generalization is associated with a certain measure on G. Let ν be a positive measure on G, and assume there exists a constant C so that

$$\nu(B(x,\rho)) \leq C\rho^m. \tag{2.1}$$

Then for fixed p, $0 < p \leq 1$, and a function f defined on $G \times (0,\infty)$, we define another functional $W_p(f)$, for $x \in G$, by

$$W_p(f)(x) = \left\{ \int_{\Gamma(x)} |f(y,t)|^2 \frac{t^{2(n-m)/p}}{t^{n+1}} d\mu(y)dt \right\}^{1/2}.$$

Note that $W_p(f)$ coincides with $A(f)$ when $m = n$.

As usual, throughout this paper C will denote a constant not necessarily the same at each occurrence.

3. Main result

LEMMA 2. *Suppose ν is a positive measure on G satisfying the condition (2.1). Let $a(y,t)$ be a T_2^p-atom supported in the tent $T(B)$ of a ball B having radius $\rho > 0$. Then for $0 < p \leq 1$, there exists a constant C_p so that*

$$\int_G W_p(a)^p(x)d\nu(x) \leq C_p.$$

Proof. Let $\chi_{B(y,t)}$ be the characteristic function of the ball $B(y,t)$ of radius t centered at y. Then

(3.1)

$$\begin{aligned}&\int_G W_p(a)^2(x)d\nu(x)\\&=\int_G\left\{\int_{\Gamma(x)}|a(y,t)|^2\frac{t^{2(n-m)/p}}{t^{n+1}}d\mu(y)dt\right\}d\nu(x)\\&=\int_G\left\{\int_{G\times(0,\infty)}|a(y,t)|^2\frac{t^{2(n-m)/p}}{t^{n+1}}\chi_{B(y,t)}(x)d\mu(y)dt\right\}d\nu(x)\\&\leq C\int_{G\times(0,\infty)}|a(y,t)|^2\frac{t^{m+2(n-m)/p}}{t^{n+1}}d\mu(y)dt,\end{aligned}$$

since

$$\int_G\chi_{B(y,t)}(x)d\nu(x)\leq Ct^m.$$

Now it is true that $t\leq 2\rho$ for $(y,t)\in T(B)$ and for small $\rho>0$. Therefore the last integral of (3.1) is less than

$$\rho^{(n-m)(2/p-1)}\int_{T(B)}|a(y,t)|^2\frac{d\mu(y)dt}{t},$$

since $a(y,t)$ is supported in $T(B)$. Thus by the definition of the atom, we have

$$\int_G W_p(a)^2d\nu(x)\leq C[\nu(B)]^{1-2/p}. \tag{3.2}$$

Note that $1-2/p<0$ is used in inequality (3.2). Finally Hölder's inequality and the inequality (3.2) give that

$$\begin{aligned}&\int_G W_p(a)^p(x)d\nu(x)\\&\leq\left\{\int_G W_p(a)^2(x)d\nu(x)\right\}^{p/2}\left\{\int_G\chi_B(x)^{2/(2-p)}d\nu(x)\right\}^{(2-p)/2}\\&\leq C_p\end{aligned}$$

for some constant C_p. The proof is therefore complete. ♮

THEOREM 3. *Suppose ν is a positive measure on G satisfying the condition* (2.1). *Let $0 < p \leq 1$, then there exists a constant C_p so that*

$$||W_p(f)||_{L^p(d\nu)} \leq C_p ||A(f)||_{L^p(d\mu)}$$

whenever $f \in T_2^p(G \times (0, \infty))$.

Proof. Let $f \in T_2^p(G \times (0,\infty))$ for $0 < p \leq 1$. Then by Lemma 1, f has an atomic decomposition

$$|f(x,t)| \leq \sum_{j=1}^{\infty} \lambda_j a_j(x,t), \text{ and } \sum_{j=1}^{\infty} \lambda_j^p \leq C||A(f)||^p_{L^p(d\mu)}.$$

Replacing $|f(x,t)|$ by its majorant $\sum_{j=1}^{\infty} \lambda_j a_j(x,t)$, we get that

$$\begin{aligned}
W_p(f)^2(x) &\leq \int_{\Gamma(x)} \Big[\sum_{j=1}^{\infty} \lambda_j a_j(y,t)\Big]^2 \frac{t^{2(n-m)/p}}{t^{n+1}} d\mu(y)dt \\
&= \sum_{i,j} \int_{\Gamma(x)} \lambda_i \lambda_j a_i(y,t) a_j(y,t) \frac{t^{2(n-m)/p}}{t^{n+1}} d\mu(y)dt \\
&\leq \sum_{i,j} \left\{ \int_{\Gamma(x)} (\lambda_i a_i(y,t))^2 \frac{t^{2(n-m)/p}}{t^{n+1}} d\mu(y)dt \right\}^{1/2} \\
&\qquad \times \left\{ \int_{\Gamma(x)} (\lambda_j a_j(y,t))^2 \frac{t^{2(n-m)/p}}{t^{n+1}} d\mu(y)dt \right\}^{1/2} \\
&= \sum_{i,j} \lambda_i \lambda_j W_p(a_i)(x) W_p(a_j)(x) \\
&= \Big[\sum_{i=1}^{\infty} \lambda_i W_p(a_i)(x)\Big]^2.
\end{aligned}$$

Thus

$$W_p(f)^p(x) \leq [\sum_{j=1}^{\infty} \lambda_j W_p(a_j)(x)]^p. \tag{3.3}$$

Integrate both sides of (3.3) with respect to $d\nu(x)$. Then we get that

$$\begin{aligned}\int_G W_p(f)^p(x)d\nu(x) &\leq \int_G \sum_{i=1}^{\infty} \lambda_i^p W_p(a_i)^p(x)d\nu(x) \\ &\leq \sum_{i=1}^{\infty} \lambda_i^p \int_G W_p(a_i)^p(x)d\nu(x) \\ &\leq C_p \sum_{i=1}^{\infty} \lambda_i^p \qquad \text{(by Lemma 2)} \\ &\leq C_p \|A(f)\|_{L^p(d\nu)}^p,\end{aligned}$$

and this completes the proof. ♯

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Department of Mathematics, Kyungpook National University, Taegu 702-701, Korea