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ON SUBMANIFOLDS OF A SPHERE WITH BOUNDED SECOND FUNDAMENTAL FORM

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1. Introduction

Let $S^{n+p}(c)$ be the $(n+p)$–dimensional Euclidean sphere of constant curva ture c and let M be an n-dimensional compact minimal submanifold isometric ally immersed in $S^{n+p}(c)$. Let A_ξ be the second fundamental form of M in the direction of a normal ξ and T the tensor defined by $T(\xi,\eta) = \text{trace} A_\xi A_\eta$.

Recently, Montiel, Ros and Urbano [M2] proved the following: Let M be an n-dimensional compact minimal submanifold isometrically immersed in $S^{n+p}(c)$. Let h be the second fundamental form of M in $S^{n+p}(c)$. If M is Einstein, $T = k <\ ,\ >$ and

$$|h|^2 \leq \frac{np(n+2)}{2(n+p+2)}c,$$

then M is isotropic and has parallel second fundamental form, where $<\ ,\ >$ is the Riemannian metric.

We'd like to consider the following problem:

Let M be an n-dimensional compact minimal submanifold isometrically immersed in $S^{n+p}(c)$. Then if $|h|^2 \leq \frac{np(n+2)}{2(n+p+2)}c$ and $T = k <\ ,\ >$, th en is the second fundamental form parallel ?

With respect to this problem Xia [X] showed: Let M be an n- dimensional compact minimal submanifold isometrically immersed in $S^{n+p}(c)$. Then

$$S \geq (n-1)c - \frac{p(n+2)}{2(n+p+2)}c \text{ and } T = k <\ ,\ >$$

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if and only if one of the following conditions is satisfied: A) $S = (n-1)c$ and M is totally geodesic, B) $S = (n-1)c - \frac{p(n+2)}{2(n+p+2)}c$ and M is isotropic and has parallel second fundamental form.

Using the result of Sakamoto [S], we know that M which is isotropic with parallel second fundamental form is a compact rank one symmetric space. Hence if the immersion ψ of M into $S^{n+p}(c)$ is full, then ψ is one of the following standard ones (See §2): $S^n(c) \to S^n(c); PR^2(\frac{1}{3}c) \to S^4(c); S^2(\frac{1}{3}c) \to S^4(c); CP^2(c) \to S^7(c) : QP^2(\frac{3}{4}c) \to S^{13}(c)$; $CayP^2(\frac{4}{3}c) \to S^{25}(c)$.

The purpose of this paper is to prove under the assumption of $|h(v,v)|^2 \leq \frac{p}{n+p+2}c$ for any $v \in UM$ and $T = k < , >$ the above result remains true.

THEOREM. *Let M be an n-dimensional compact minimal submanifold isometrically immersed in $S^{n+p}(c)$ and ψ the immersion. Let A_ξ be the second fundamental form of M in the direction of a normal ξ and T the tensor defined by $T(\xi,\eta) = \text{trace} A_\xi A_\eta$ for any normal ξ, η. Then*

$$|h(v,v)|^2 \leq \frac{p}{n+p+2}c \textit{ and } T = k < , >$$

if and only if one of the following conditions is satisfied:

A) $|h(v,v)|^2 \equiv 0$ and M is totally geodesic,

B) $|h(v,v)|^2 = \frac{p}{n+p+2}c$ and M is isotropic and has parallel second fundamental form, where $< , >$ is the Riemannian metric. Hence if ψ is full, then ψ is one of the following standard ones: $S^n(c) \to S^n(c); PR^2(\frac{1}{3}c) \to S^4(c); S^2(\frac{1}{3}c) \to S^4(c); CP^2(c) \to S^7(c) : QP^2(\frac{3}{4}c) \to S^{13}(c); CayP^2(\frac{4}{3}c) \to S^{25}(c)$.

REMARK. With respect to Theorem Gauchman [G] proved the similar result under the assumption of $|h(v,v)|^2 \leq \frac{1}{3}c$ without the assumption on T.

2. Preliminaries

Let M be a compact Riemannian manifold, UM its unit tangent bundle, and UM_x the fibre of UM over a point x of M. We denote by

dx, dv and dv_x denote the canonical measures on M, UM and UM_x respec tively.

For any continuous function $f : UM \to R$, we have

$$\int_{UM} f dv = \int_M (\int_{UM_x} f dv_x) dx.$$

If T is a k-covariant tensor on M and ∇T is its covariant derivati ve, then we have

$$\int_{UM} \{\sum_{i=1}^{n} (\nabla T)(e_i, e_i, v, \cdots, v)\} dv = 0,$$

where $e_1, \cdots, e_n$ is an orthonormal basis of $T_x M, x \in M$.

Now, we suppose that M is isometrically immersed in an $(n+p)$-dimensional R iemannian manifold $\overline{M}$. We denote by $< , >$ the metric of $\overline{M}$ as well as that induced on M. If h is the second fundamental form of the immersion and A_ξ the Weingarten endomorphism associated a normal vector ξ, we define

$$T : T_x^\perp M \times T_x^\perp M \to R$$

by the expression

$$T(\xi, \eta) = \text{trace} A_\xi A_\eta,$$

where $T_x^\perp M$ is the normal space to M at x. Then T is a symmetr ic bilinear map.

Let X, Y, Z and W denote the tangent vector fields on M. Then, if ∇h and $\nabla^2 h$ denote the first and second covariant derivatives of h, respectively, one has that ∇h is symmetric and $\nabla^2 h$ satisfies the following relation:

$$(2.1) \quad \begin{aligned} (\nabla^2 h)(X, Y, Z, W) = (\nabla^2 h)(Y, X, Z, W) + R^\perp(X, Y) h(Z, W) \\ - h(R(X, Y)Z, W) - h(Z, R(X, Y)W), \end{aligned}$$

where $R^\perp$ and R are the curvature operators of the normal and tangent bundles over M, respectively.

Now let $v \in UM_x, x \in M$. If $e_2, \ldots, e_n$ are orthonormal vecto rs in UM_x orthogonal to v, then we can consider $\{e_2, \ldots, e_n\}$ as an orthonormal basis of $T_v(UM_x)$. We remark that $\{v = e_1. e_2, \ldots, e_n\}$ is an orthonormal basis of T_xM. If we denote the Laplacian of $UM_x \cong S^{n-1}$ by Δ, then $\Delta f = e_2 e_2 f + \cdots + e_n e_n f$, where f is a differentiable function on UM_x.

Define a function f_1 on $UM_x, x \in M$, by

$$f_1(v) = |A_{h(v,v)}v|^2 = \sum_{i=1}^{n} < h(v,v), h(v,e_i) >^2 .$$

Using the minimality of M we can prove that

(2.2)

$$\begin{aligned}(\Delta f_1)(v) &= -6(n+4) f_1(v) \\ &+ 8 \sum_{i=1}^{n} < A_{h(v,v)}v, A_{h(v,e_i)}e_i > + 8 \sum_{i=1}^{n} < A_{h(v,v)}e_i, A_{h(v,e_i)}v > \\ &+ 8 \sum_{i=1}^{n} < A_{h(v,e_i)}v, A_{h(v,e_i)}v > \\ &+ 2 \sum_{i=1}^{n} < A_{h(v,v)}e_i, A_{h(v,v)}e_i > .\end{aligned}$$

Similarly, define $f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9$ and f_{10} by

$$f_2(v) = \sum_{i=1}^{n} < A_{h(v,e_i)}v, A_{h(v,e_i)}v >,$$

$$f_3(v) = \sum_{i=1}^{n} < A_{h(v,e_i)}v, A_{h(v,v)}e_i >,$$

$$f_4(v) = \sum_{i,j=1}^{n} < A_{h(e_j,e_i)}e_j, A_{h(v,v)}e_i >,$$

$$f_5(v) = \sum_{i=1}^{n} < A_{h(v,v)}v, A_{h(v,e_i)}e_i >,$$

$$f_6(v) = \sum_{i,j=1}^{n} < A_{h(e_j,e_i)}e_j, A_{h(v,e_i)}v >,$$

$$f_7(v) = \sum_{i,j=1}^{n} < A_{h(e_i,v)}e_i, A_{h(v,e_j)}e_j >,$$

$$f_8(v) = \sum_{i=1}^{n} < A_{h(v,v)}e_i, A_{h(v,v)}e_i >,$$

$$f_9(v) = |h(v,v)|^4,$$

$$f_{10}(v) = |h(v,v)|^2,$$

respectively. Then we know that

$$(2.3) \quad \begin{aligned} (\Delta f_2)(v) = & -4(n+2)f_2(v) + 4f_6(v) \\ & + 4\sum_{i,j=1}^{n} < A_{h(e_j,e_i)}v, A_{h(v,e_i)}e_j > \\ & + 2\sum_{i,j=1}^{n} < A_{h(e_j,e_i)}v, A_{h(e_j,e_i)}v > \\ & + 2\sum_{i,j=1}^{n} < A_{h(v,e_i)}e_j, A_{h(v,e_i)}e_j >, \end{aligned}$$

$$(2.4) \quad \begin{aligned} (\Delta f_3)(v) = & -4(n+2)f_3(v) + 2f_4(v) \\ & + 4\sum_{i,j=1}^{n} < A_{h(e_j,e_i)}v, A_{h(e_j,v)}e_i > \\ & + 4\sum_{i,j=1}^{n} < A_{h(v,e_i)}e_j, A_{h(e_j,v)}e_i >. \end{aligned}$$

$$(2.5) \qquad (\Delta f_4)(v) = -2nf_4(v).$$

$$(2.6) \qquad (\Delta f_5)(v) = -4(n+2)f_5(v) + 4f_6(v) + 4f_7(v) + 2f_4(v),$$

$$(2.7) \qquad (\Delta f_6)(v) = -2nf_6(v) + 2\sum_{i,j,k=1}^{n} < A_{h(e_j,e_i)}e_j, A_{h(e_k,e_i)}e_k >.$$

$$(2.8)\qquad (\Delta f_7)(v) = -2nf_7(v) + 2\sum_{i,j,k=1}^{n} < A_{h(e_j,e_i)}e_j, A_{h(e_k,e_i)}e_k >,$$

$$(2.9)\qquad (\Delta f_8)(v) = -4(n+2)f_8(v) + 8\sum_{i,j=1}^{n} < A_{h(e_j,v)}e_i, A_{h(e_j,v)}e_i >,$$

(2.10)

$$\begin{aligned}(\Delta f_9)(v) = & -8(n+6)f_9(v) + 32f_1(v) \\ & + 16\sum_{i=1}^{n} < A_{h(v,e_i)}e_i, v > |h(v,v)|^2 \\ \geq & -8(n+2)f_1(v) + 16\sum_{i=1}^{n} < A_{h(v,e_i)}e_i, v > |h(v,v)|^2.\end{aligned}$$

Then we have the following (See [M1] and [M2]):

LEMMA 1. *Let M be an n-dimensional compact minimal submanifold iso metrically immersed in $\overline{M}$. Then for all $x \in M$, we have*

$$(2.11)\qquad \int_{UM_x} |h(v,v)|^2 dv_x = \frac{2}{n+2}\int_{UM_x}\sum_{i=1}^{n} < A_{h(v,e_i)}e_i, v > dv_x,$$

$$(2.12)\qquad \int_{UM_x}\sum_{i=1}^{n} < A_{h(v,e_i)}e_i, v > dv_x = \frac{1}{n}\int_{UM_x} |h|^2 dv_x,$$

$$(2.13)\qquad \begin{aligned}&\int_{UM_x} |A_{h(v,v)}v|^2 dv_x \\ &\geq \frac{2}{n+2}\int_{UM_x}\sum_{i=1}^{n} < A_{h(v,e_i)}e_i, A_{h(v,v)}v > dv_x,\end{aligned}$$

$$(2.14)\qquad \begin{aligned}&\int_{UM_x} |A_{h(v,v)}v|^2 dv_x \\ &\geq \frac{2}{n+2}\int_{UM_x}\sum_{i=1}^{n} < A_{h(v,e_i)}e_i, v > |h(v,v)|^2 dv_x,\end{aligned}$$

where $v \in UM_x$ and $\{e_i\}_{i=1}^n$ is an orthonormal basis of the ta ngent space T_xM to M at x.

Since

$$\frac{1}{2}\sum_{i=1}^n(\nabla^2 f_{10})(e_i,e_i,v) = \sum_{i=1}^n < (nabla^2 h)(e_i,e_i,v,v), h(v,v) > + \sum_{i=1}^n < (\nabla h)(e_i,v,v),(\nabla h)(e_i,v,v) >,$$

we have

LEMMA 2. *Let M be an n-dimensional minimal submanifold isometrical ly immersed in $S^{n+p}(c)$. Then for $v \in UM_x$ we have*
(2.15)

$$\begin{aligned} &\frac{1}{2}\sum_{i=1}^n(\nabla^2 f_{10})(e_i,e_i,v) \\ &= \sum_{i=1}^n |(\nabla h)(e_i,v,v)|^2 + nc|h(v,v)|^2 \\ &+ 2\sum_{i=1}^n < A_{h(v,v)}e_i, A_{h(e_i,v)}v > -2\sum_{i=1}^n < A_{h(v,e_i)}e_i, A_{h(v,v)}v > \\ &- \sum_{i=1}^n < A_{h(v,v)}e_i, A_{h(v,v)}e_i > . \end{aligned}$$

Also, as for $v \in UM_x, x \in M$ the following equations hold:

$$\sum_{i,j=1}^n < A_{h(e_j,e_i)}v, A_{h(e_j,v)}e_i >= \sum_{i,j=1}^n < A_{h(v,e_i)}e_j, A_{h(v,e_j)}e_i >,$$

$$\sum_{i,j=1}^n < A_{h(e_j,e_i)}v, A_{h(e_j,e_i)}v >= \sum_{i,j=1}^n < A_{h(v,e_i)}e_j, A_{h(v,e_i)}e_j >,$$

we get

LEMMA 3. *Let M be an n-dimensional minimal submanifold isometrically immersed in $S^{n+p}(c)$. Then for $v \in UM_x$ we have*
(2.16)

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_{10})(e_i, e_i, v) - \frac{1}{6}(\Delta f_1)(v) - \frac{1}{3(n+2)}(\Delta f_2)(v) \\
&+ \frac{1}{6(n+2)}(\Delta f_3)(v) + \frac{1}{3n(n+2)}(\Delta f_4)(v) + \frac{1}{6(n+2)}(\Delta f_5)(v) \\
&- \frac{1}{3n(n+2)}(\Delta f_6)(v) + \frac{1}{3n(n+2)}(\Delta f_7)(v) + \frac{1}{6(n+2)}(\Delta f_8)(v) \\
= &\sum_{i=1}^{n} |(\nabla h)(e_i, v, v)|^2 + ncf_{10}(v) + (n+4)f_1(v) - 4f_5(v) - 2f_8(v).
\end{aligned}$$

Now, We will consider the condition $T = k < , >$. Let M be a compact Rieman nian manifold and $0 = \lambda_0 < \lambda_1 < . \; .$ the eigenvalue sequen ce of its Laplacian Δ. Let $\phi : M \to E^m$ be an isometric immersio n in the Euclidean space. Then we have the L^2-decomosition $\phi = \sum_{u \geq 0} \phi_u, u \in N$, where $\phi_u : M \to E^m$ is a differentiable mapp ing satisfying $\Delta\phi_u = -\lambda_u\phi_u$. Note that ϕ_0 is a constant map.

Let $u_1, u_2 \in N, 1 \leq u_1 \leq u_2$. We say that *the immersi on ϕ is by the $\{u_1, u_2\}$-th eigenfunctions* if $\phi_u = 0$ for all $u \in N, u \neq 0, u_1$ or u_2. If $u_1 = u_2$ w e say that *the immersion is by the u_1-th eigenfunction.*

We say that ϕ is *a standard immersion by the u-th eigenfunctio n* if $\phi = k(f_1, \ldots, f_{m_u})$ where k is a real constant and $f_1, \ldots, f_{m_u}$ is an orthonormal basis of the λ_u-eigenspace.

Consider on the space $SM(m) = \{P \in gl\,(m, R) : P = {}^tP\}$ of the symmetric matrices of order m with the metric g given by

$$g(P, Q) = \frac{c}{2}\text{trace}PQ \quad \text{for all} \quad P, Q \in SM(m).$$

Then the mapping $f : S^m(c) \to SM(m+1)$ given by $f(x) = {}^txx$ is an isometric immersion by the second eigenfunction.

Let $\psi : M \to S^{n+p}(c)$ be a full and minimal isometric immersion of an n-dimensional Riemannian manifold M. If ψ is considered as an immersio n in E^{n+p+1}, then ψ is by the u-th eigenfunction for some

$u \in N$ with $\lambda_u = nc$. We consider the isometric immersion $\phi = f \circ \psi$: $M \to SM(n+p+1)$. If M is a compact rank one symmetric space and ψ is the first standard immersion, then $\phi = f \circ \psi$ is an immersi on by the $\{1, 2\}$-nd eigenfunctions. Then

LEMMA 4. ([M2]) *Let $\psi : M \to S^{n+p}(c)$ be a full and min imal isometric immersion and $\phi : M \to SM(n+p+1)$ the associated immersi on. Then ϕ is an immersion by the $\{u_1, u_2\}$-th eigenfunctions i f and only if M is Einstein and $T = k < , >$. Moreover, in this case ϕ_0 is proportional to the identity matrix in $SM(n+p+1)$.*

LEMMA 5. ([M2]) *Let $\psi : M \to S^{n+p}(c)$ be a full and minim al isometric immersion by the first eigenfunctions i.e., $\lambda_1 = nc$. If*

$$\int_M x_i x_j dp = \alpha \delta_{ij}$$

where α is a real number and $x_0, \ldots, x_{n+p}$ are the coordinate f unctions of ψ, then

$$\int_M |h|^2 dp \geq \frac{(n+2)p\lambda_1}{2(n+p+2)} \mathrm{vol}(M).$$

The equality holds if and only if the immersion $\phi = f \circ \psi$ is by the $\{1, u\}$-th eigenfunctions for some $u \in N$.

3. Proof of theorem

Integrating (2.16) over UM_x and substituting (2.13), we obtain

$$\begin{aligned}
0 \geq & \int_{UM_x} \sum_{i=1}^{n} |(\nabla h)(e_i, v, v)|^2 dv_x + nc \in t_{UM_x} |h(v, v)|^2 dv_x \\
(3.1) \qquad & - n \int_{UM_x} |A_{h(v,v)} v|^2 dv_x \\
& - 2 \int_{UM_x} \sum_{i=1}^{n} < A_{h(v,v)} e_i, A_{h(v,v)} e_i > dv_x.
\end{aligned}$$

For any v in UM_x, we can put $h(v,v) = |h(v,v)|\xi$ for some unit vector ξ normal to M. Since $|h(v,v)|^2 \leq \frac{p}{n+p+2}c$ for any $v \in UM_x$, we have by Schwarz's inequality,

$$|A_\xi u|^2 \leq (\text{maximum eigenvalue of } A_\xi)^2 \leq \frac{p}{n+p+2}c \text{ for any } u \in UM_x.$$

Hence

$$|A_{h(v,v)}v|^2 \leq \frac{p}{n+p+2}c|h(v,v)|^2. \tag{3.2}$$

On the other hand, from the assumption of $T = k <\ ,\ >$, taking the trace, we h ave $k = \frac{|h|^2}{p}$. From (2.11) and (2.12) we have
(3.3)

$$\int_{UM_x} \sum_{i=1}^{n} < A_{h(v,v)}e_i, A_{h(v,v)}e_i > dv_x = \frac{1}{p}\int_{UM_x} |h|^2|h(v,v)|^2 dv_x$$

$$\leq \frac{1}{n+p+2}c\int_{UM_x} |h|^2 dv_x = \frac{n(n+2)}{2(n+p+2)}c\int_{UM_x} |h(v,v)|^2 dv_x.$$

Combining (3.1), (3.2) and (3.3), we have

$$\begin{aligned}(3.4)\quad 0 \geq & \int_{UM_x} \sum_{i=1}^{n} |(\nabla h)(e_i,v,v)|^2 dv_x \\ & + (nc - \frac{np}{n+p+2}c - \frac{n(n+2)}{n+p+2}c)\int_{UM_x} |h(v,v)|^2 dv_x \geq 0.\end{aligned}$$

Hence h is parallel. Also, the equalities of (3.2) and $|h(v,v)|^2 \leq \frac{p}{n+p+2}c$ hold. Thus we get

$$\begin{aligned}\int_{UM_x} |A_{h(v,v)}v|^2 dv_x &= \frac{p}{n+p+2}c\int_{UM_x} |h(v,v)|^2 dv_x \\ &= \frac{2p}{(n+2)(n+p+2)}c\int_{UM_x} \sum_{i=1}^{n} < A_{h(v,e_i)}e_i, v > dv_x \\ &= \frac{2}{n+2}\int_{UM_x} \sum_{i=1}^{n} < A_{h(v,e_i)}e_i, v > |h(v,v)|^2 dv_x.\end{aligned}$$

Therefore (2.10) and (2.16) show that $|A_{h(v,v)}v|^2 = |h(v,v)|^4$, i,e., M is isotropic at x.

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