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HYPERBOLIC HOMEOMORPHISMS

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1. Introduction

In [6], we introduce a hyperbolic homeomorphism on a compact metrizable space and show that a hyperbolic homeomorphism is topologically stable. The purpose of this paper is to study a necessary and sufficient condition for a homeomorphism to be hyperbolic. We get the following theorem.

THEOREM. *Let X be a compact metrizable space. A homeomorphism $f : X \to X$ is hyperbolic if and only if f is expansive and has the pseudo orbits tracing property.*

2. Preliminaries

Let X be a compact metric space with a metric d and $f : X \to X$ a homeomorphism. Recall that f is expansive if there exists a constant $\epsilon > 0$ such that

$$d(f^n(x), f^n(y)) \leq \epsilon \quad \text{for all} \quad n \in Z \quad \text{implies} \quad x = y.$$

The constant ϵ is called expansive constant for f. This property is independent of the metric chosen for X. Recall that a sequence $(x_i)_{i=m}^{n}$, $-\infty \leq m \leq 0 \leq n \leq \infty$, in X is a δ-pseudo orbit if

$$d(f(x_i), x_{i+1}) \leq \delta \quad \text{for all} \quad m - 1 < i < n.$$

We say that f has the pseudo orbits tracing property if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that any δ-pseudo orbit

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$(x_i)_{i=m}^{n}, -\infty \le m \le 0 \le n \le \infty$, is ε-traced by some point x of X, that is,

$$d(f^i(x), x_i) \le \varepsilon \quad \text{for all} \quad m-1 < i < n+1.$$

This property is independent of the metric chosen for X. Let $x \in X$ and $\varepsilon > 0$. We define the stable and unstable set of size ε at x:

$$W_d^s(x,\varepsilon) = \{y \in X | d(f^n(x), f^n(y)) \le \varepsilon \quad \text{for all} \quad n \ge 0\},$$
$$W_d^u(x,\varepsilon) = \{y \in X | d(f^n(x), f^n(y)) \le \varepsilon \quad \text{for all} \quad n \le 0\}.$$

We define a closed neighborhood $B_d(\varepsilon)$ of the diagonal Δ of X^2 by

$$B_d(\varepsilon) = \{(x,y) \in X^2 | d(x,y) \le \varepsilon\}.$$

DEFINITION 2.1. Let X be a compact metrizable space. A homeomorphism $f : X \to X$ is hyperbolic if there exist a metric d compatible with the topology of X, constants $\varepsilon_0 > 0, \delta_0 > 0, c > 0, 0 < \lambda < 1$, and a continuous map $[\ ,\ ] : B_d(\delta_0) \to X$ such that the following three conditions hold.

(1) if $y \in W_d^s(x,\varepsilon_0)$, then $d(f^n(x), f^n(y)) \le c\lambda^n d(x,y)$ for all $n \ge 0$,

(2) if $y \in W_d^u(x,\varepsilon_0)$, then $d(f^n(x), f^n(y)) \le c\lambda^{-n} d(x,y)$ for all $n \le 0$,

(3) if $d(x,y) \le \delta_0$, then $W_d^s(x,\varepsilon_0) \cap W_d^u(y,\varepsilon_0) = \{[x,y]\}$.

The following lemma is due to Frink ([2], [3]).

LEMMA 2.1. *Let $(W_n)_{n=0}^{\infty}$ be a nested sequence of symmetric neighborhoods of the diagonal Δ of X^2 with $W_0 = X^2$ such that*

$$W_{n+1} \circ W_{n+1} \circ W_{n+1} \subset W_n \quad \text{for all} \quad n \ge 0,$$
$$\bigcap_{n=0}^{\infty} W_n = \Delta.$$

Then there exists a metric d compatible with the topology of X such that

$$W_n \subset B_d(\frac{1}{2^n}) \subset W_{n-1} \quad \text{for all} \quad n \ge 1.$$

3. Proof of the theorem

Suppose that f is hyperbolic with respect to a metric d. Let $e = \min\{\varepsilon_0, \delta_0\}$. If $d(f^n(x), f^n(y)) \leq e$ for all $n \in \mathbb{Z}$, then $x, y \in W^s_d(x, \varepsilon_0) \cap W^u_d(y, \varepsilon_0) = \{[x, y]\}$. Thus $x = y$. Hence f is expansive.

To show that f has the pseudo orbits tracing property the following lemma is needed.

LEMMA 3.1. *Assume that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that every δ-pseudo orbit $(x_i)_{i=0}^n, 0 \leq n < \infty$, is ε-traced by some point of X. Then f has the pseudo orbits tracing property.*

Proof. We first show that every δ-pseudo orbit $(x_i)_{i=m}^n, -\infty < m \leq 0 \leq n < \infty$, can be ε-traced by some point of X. Let $y_j = x_{j+m}, 0 \leq j \leq n - m$. Then $(y_j)_{j=0}^{n-m}$ is also a δ-pseudo orbit. Thus $(y_j)_{j=0}^{n-m}$ is ε-traced by some point y of X. Let $x = f^{-m}(y)$. For $m \leq i \leq n$, we have

$$\begin{aligned} d(f^i(x), x_i) &= d(f^{i-m}(y), x_i) \\ &= d(f^j(y), x_{j+m}) = d(f^j(y), y_j) \leq \varepsilon. \end{aligned}$$

Thus $(x_j)_{i=m}^n$ is ε-traced by x.

We now show that f has the pseudo orbits tracing property. Let $(x_i)_{i=-\infty}^{\infty}$ be a δ-pseudo orbit. For each $n > 0$, the δ-pseudo orbit $(x_i)_{i=-n}^n$ is ε-traced by some point y_n of X. Since X is compact, the sequence (y_n) has a convergent subsequence (y_{n_j}). Let $y_{n_j} \to y$. For each $i \in \mathbb{Z}$, since $f^i(y_{n_j}) \to f^i(y)$, we can choose j such that

$$d(f^i(y), f^i(y_{n_j})) \leq \varepsilon \quad \text{and} \quad -n_j \leq i \leq n_j$$

Then we have

$$\begin{aligned} d(f^i(y), x_i) &\leq d(f^i(y), f^i(y_{n_j})) + d(f^i(y_{n_j}), x_i) \\ &\leq 2\varepsilon. \end{aligned}$$

Thus $(x_i)_{i=-\infty}^{\infty}$ is 2ε-traced by y. Hence f has the pseudo orbits tracing property.

Let us show that f has the pseudo orbit tracing property. Given any $\varepsilon > 0$, let $0 < p \leq \min\{c\varepsilon, \frac{(1-\lambda)\varepsilon}{2c}\}, 0 < \delta_1 \leq \min\{\varepsilon_0, \delta_0, \frac{p}{c}\}$. Then

$$d(x,y) \leq \delta_1 \quad \text{implies} \quad W_d^s(x,p) \cap W_d^u(y,p) = \{[x,y]\}.$$

Take $m > 0$ such that $\lambda^m < \frac{\delta_1}{2cp}$ and let $\delta_2 = \frac{\delta_1}{2m}$. There exists a $\delta > 0$ such that

$$d(x,y) \leq \delta \quad \text{implies} \quad d(f^i(x), f^i(y)) \leq \delta_2 \quad \text{for all} \quad 0 \leq i < m.$$

Given any δ-pseudo orbit $(x_i)_{i=0}^n, 0 \leq n < \infty$, choose $k > 0$ such that $n < km$ and let

$$y_i = \begin{cases} x_i & \text{for} \quad 0 \leq i \leq n \\ f^{i-n}(x_n) & \text{for} \quad n \leq i \leq km. \end{cases}$$

Then $(y_i)_{i=0}^{km}$ is a δ-pseudo orbit. Let $1 \leq j \leq m$. For any i,

$$\begin{aligned} d(f^j(y_i), y_{i+j}) &\leq \sum_{s=0}^{j-1} d(f^{j-s}(y_{i+s}), f^{j-s-1}(y_{i+s+1})) \\ &\leq j\delta_2 \leq \frac{\delta_1}{2}. \end{aligned}$$

Let $z_j = [y_{jm}, f^m(z_{j-1})], z_0 = y_0$. Let $w = f^{km}(z_k)$. For $(i-1)m \leq j < im, 1 \leq i \leq k$, we have

$$\begin{aligned} d(f^j(w), y_j) \leq\ & d(f^{j-km}(z_k), f^{j-(i-1)m}(z_{i-1})) \\ &+ d(f^{j-(i-1)m}(z_{i-1}), f^{j-(i-1)m}(y_{(i-1)m})) \\ &+ d(f^{j-(i-1)m}(y_{(i-1)m}), y_j), \end{aligned}$$

$$\begin{aligned} & d(f^{j-km}(z_k), f^{j-(i-1)m}(z_{i-1})) \\ \leq & \sum_{s=0}^{k-i} d(f^{j-(k-s)m}(z_{k-s}), f^{j-(k-(i-1)m)}(z_{k-s-1})) \\ \leq & \sum_{s=0}^{k-i} cp\lambda^{(k-s)m-j}, \end{aligned}$$

$$d(f^{j-(i-1)m}(z_{i-1}), f^{j-(i-1)m}(y_{(i-1)m})) \leq cp\lambda^{j-(i-1)m} \leq cp$$
$$d(f^{j-(i-1)m}(y_{(i-1)m}), y_j) \leq \frac{\delta_1}{2}.$$

Thus $d(f^j(w), y_j) \leq cp\sum_{s=0}^{\infty} \lambda^s + \frac{\delta_1}{2} \leq \varepsilon$. Hence $(x_i)_{i=0}^n$ is ε-traced by w. By Lemma 3.1, f has the pseudo orbits tracing property.

Suppose that f is expansive and has the pseudo orbits tracing property with respect to metric d. To show that f is hyperbolic we need some lemmas.

We define a nested sequence of closed symmetric neighborhoods of the diagonal Δ of X^2 as follows. Set $V_0 = X^2$ and for $n \geq 1$ define the set V_n as follows:

$$V_n = \left\{(x, y) \in X^2 \mid d(f^i(x), f^i(y)) \leq e \quad \text{for} \quad |i| \leq n-1\right\}.$$

LEMMA 3.2. *For each $\varepsilon > 0$ there exists $m > 0$ such that $V_m \subset B_d(\varepsilon)$.*

Proof. Suppose that Lemma 3.2 does not hold. There exists $\varepsilon_0 > 0$ such that $V_n \not\subset B_d(\varepsilon)$ for all $n > 0$. For each $n > 0$ there exists $(x_n, y_n) \in V_n - B_d(\varepsilon_0)$. Since X^2 is compact, the sequence $((x_n, y_n))$ has a convergent subsequence. Let $(x_n, y_n) \to (x, y)$. For any $n \in \mathbb{Z}$, take $m > |n|$, then

$$d(f^n(x_m), f^n(y_m)) \leq e \quad \text{because} \quad (x_m, y_m) \in V_m.$$

As $m \to \infty, d(f^n(x), f^n(y)) \leq e$. Thus $x = y$. Since $d(x_n, y_n) > \varepsilon_0$ for all $n > 0$, take $n \to \infty, d(x, y) \geq \varepsilon_0$. We have a contradiction . Thus Lemma 3.2 holds.

LEMMA 3.3. $\bigcap_{n=0}^{\infty} V_n = \Delta$.

Proof. It is obvious that $\Delta \subset \bigcap_{n=0}^{\infty} V_n$. Let $(x, y) \in \bigcap_{n=0}^{\infty} V_n$. For any $n \in \mathbb{Z}$, since $(x, y) \in V_{|n|+1}, d(f^n(x), f^n(y)) \leq e$. Thus $x = y$ so $(x, y) \in \Delta$. Hence $\bigcap_{n=0}^{\infty} V_n = \Delta$

By Lemma 3.2, there exists $m > 0$ such that $V_{m+1} \subset B_d(\frac{e}{3}) \subset B_d(e) = V_1$. We define a new nested sequence of closed symmetric neighborhoods of Δ as follows. Put $W_0 = V_0$ and $W_k = V_{(k-1)m+1}$.

LEMMA 3.4.
(1) $\bigcap_{k=0}^{\infty} W_k = \Delta$
(2) $W_{k+1} \circ W_{k+1} \circ W_{k+1} \subset W_k$ *for all* $k \geq 0$.

Proof. (1) It is obvious that $\Delta \subset \bigcap_{n=0}^{\infty} W_n$. Let $(x, y) \in \bigcap_{n=0}^{\infty} W_n$. For any $n \geq 0$, there exists $k > 0$ such that $n \leq (k-1)m + 1$. We have $(x, y) \in W_k = V_{(k-1)m+1} \subset V_n$. Thus $(x, y) \in \bigcap_{n=0}^{\infty} V_n = \Delta$. Hence $\bigcap_{n=1}^{\infty} W_n = \Delta$.

(2) Trivially, $W_1 \circ W_1 \circ W_1 \subset X^2 = V_0 = W_0$. Let $(x, y) \in W_2 \circ W_2 \circ W_2 = V_{m+1} \circ V_{m+1} \circ V_{m+1}$. There exist $z, w \in X$ such that $(x, z), (z, w), (w, y) \in V_{m+1} \subset B_d(\frac{e}{3})$. Since $d(x, y) \leq d(x, z) + d(z, w) + d(w, y) \leq \frac{e}{3} + \frac{e}{3} + \frac{e}{3} = e, (x, y) \in B_d(e) = V_1 = W_1$. Thus we have $W_2 \circ W_2 \circ W_2 \subset W_1$. Now suppose $(x, y) \in W_{k+1} \circ W_{k+1} \circ W_{k+1}$ for $k > 1$. Then there exist $z, w \in X$ such that $(x, z), (z, w), (w, y) \in W_{k+1} = V_{km+1}$. If (p, q) is any of those three pairs, then $d(f^i(p), f^i(q)) \leq e$ for $|i| \leq km$. If $|i| \leq (k-1)m$ and $|j| \leq m$, then $|i + j| \leq km$ so we may compute as follows:

$$d(f^j(f^i(p)), f^j(f^i(q))) = d(f^{i+j}(p), f^{i+j}(q)) \leq e.$$

Therefore, $(f^i(p), f^i(q)) \in V_{m+1} = W_2$ and hence

$$(f^i(x), f^i(y)) \in W_2 \circ W_2 \circ W_2 \subset W_1 = V_1 = B_d(e).$$

That is $d(f^i(x), f^i(y)) \leq e$ for $|i| \leq (k-1)m$. Therefore $(x, y) \in V_{(k-1)m+1} = W_k$. So $W_{k+1} \circ W_{k+1} \circ W_{k+1} \subset W_k$.

The following lemma is an immediate consequence of Lemma 2.1 and Lemma 3.4.

LEMMA 3.5. *There exists a metric D compatible with the topology of X such that*

$$B_D\left(\frac{1}{2^{n+1}}\right) \subset W_n \subset B_D\left(\frac{1}{2^n}\right) \quad \textit{for} \quad n \geq 1.$$

LEMMA 3.6.

$$f(W_d^s(x,e)\cap xV_n) = W_d^s(f(x),e)\cap f(x)V_{n+1},$$
$$f^{-1}(W_d^u(x,e)\cap xV_n) = W_d^u(f^{-1}(x),e)\cap f^{-1}(x)V_{n+1}$$

for all $n \geq 1$.

Proof. Let $y \in f(W_d^s(x,e)\cap xV_n)$. Then $f^{-1}(y) \in W_d^s(x,e)\cap xV_n$. For $i \geq 0, d(f^if(x), f^i(y)) = d(f^{i+1}(x), f^{i+1}f^{-1}(y)) \leq e$. Thus $y \in W_d^s(f(x),e)$. For $-n \leq i \leq n, d(f^if(x), f^i(y)) = d(f^{i+1}(x), f^{i+1}f^{-1}(y)) \leq e$ since $-(n-1) \leq i+1$. Thus $(f(x),y) \in V_{n+1}$. Hence $y \in W_d^s(f(x),e)\cap f(x)V_{n+1}$. Let $y \in W_d^s(f(x),e)\cap f(x)V_{n+1}$. Then $d(x,f^{-1}(y)) = d(f^{-1}f(x), f^{-1}(y)) \leq e$, and for $i \geq 1$, $d(f^i(x), f^if^{-1}(y)) = d(f^{i-1}f(x), f^{i-1}(y)) \leq e$. Thus $f^{-1}(y) \in W_d^s(x,e)$. For $-(n-1) \leq i \leq n-1, d(f^i(x), f^if^{-1}(y)) = d(f^{i-1}f(x), f^{i-1}(y)) \leq e$ since $-n \leq i-1$. Thus $(x, f^{-1}(y)) \in V_n$. Hence $f^{-1}(y) \in W_d^s(x,e)\cap xV_n$ that is $y \in f(W_d^s(x,e)\cap xV_n)$.

LEMMA 3.7. *If* $0 < D(x,y) \leq \frac{1}{4}$, *then there exists* $n \geq 2$ *such that* $(x,y) \in W_{n-1} - W_n$.

Proof. Since $x \neq y, (x,y) \in \Delta^c = (\bigcap_{n=0}^{\infty} W_n)^c = \bigcup_{n=0}^{\infty} W_n^c$. Thus there exists $n \geq 1$ such that $(x,y) \notin W_n$. Let $k = \min\{n \geq 1 \mid (x,y) \notin W_n\}$. Then $(x,y) \in W_{k-1} - W_k$. Since $(x,y) \notin W_k$ and $B_D(\frac{1}{2^{k+1}}) \subset W_k, (x,y) \notin B_D(\frac{1}{2^{k+1}})$. Thus $\frac{1}{2^{k+1}} < D(x,y) \leq \frac{1}{4}$. Hence $k \geq 2$.

Let us show that f is hyperbolic with respect to D. There exists $\varepsilon_1 > 0$ such that $D(x,y) \leq \varepsilon_1$ implies $d(x,y) \leq e$. Therefore $W_D^s(x,\varepsilon_1) \subset W_d^s(x,e)$. Let $\varepsilon_0 = \min\{\varepsilon_1, \frac{1}{4}\}$. We first show that if $y \in W_D^s(x,\varepsilon_0)$, then

$$D(f^{3m}(x), f^{3m}(y)) \leq \frac{1}{2}D(x,y).$$

If $x = y$, then this is obvious. Let $x \neq y$. Since $0 < D(x,y) \leq \varepsilon_0 \leq \frac{1}{4}$, by lemma 3.7, there exists $n \geq 2$ such that $(x,y) \in W_{n-1} - W_n$. For $i \geq 0, d(f^i(x), f^i(y)) \leq e$ since $D(f^i(x), f^i(y)) \leq \varepsilon_0 \leq \varepsilon_1$. Thus $y \in W_d^s(x,e)$. Since $(x,y) \in W_{n-1}, y \in xW_{n-1}$. Thus $y \in W_d^s(x,e)\cap$

xW_{n-1}. By Lemma 3.6,

$$\begin{aligned}
f^{3m}(y) &\in f^{3m}(W_d^s(x,e) \cap xW_{n-1}) \\
&= f^{3m}(W_d^s(x,e) \cap xV_{(n-2)m+1}) \\
&= W_d^s(f^{3m}(x),e) \cap f^{3m}(x)V_{(n+1)m+1} \\
&= W_d^s(f^{3m}(x),e) \cap f^{3m}(x)W_{n+2} \\
&\subset f^{3m}(x)W_{n+2} \subset f^{3m}(x)B_D\Big(\frac{1}{2^{n+2}}\Big).
\end{aligned}$$

Thus $(f^{3m}(x), f^{3m}(y)) \in B_D(\frac{1}{2^{n+2}})$. Therefore $D(f^{3m}(x), f^{3m}(y)) \leq \frac{1}{2^{n+2}} = \frac{1}{2} \cdot \frac{1}{2^{n+1}} < \frac{1}{2}D(x,y)$. By induction,

$$D(f^{3km}(x), f^{3km}(y)) \leq \frac{1}{2^k}D(x,y).$$

Let $\lambda = 2^{-\frac{1}{3m}}$. Then $0 < \lambda < 1$. Let us show that if $y \in W_D^s(x, \varepsilon_0)$, then $D(f^n(x), f^n(y)) \leq 8\lambda^n D(x,y)$ for $n \geq 0$. If $x = y$, then the above inequality holds. Let $x \neq y, n = 3km + i, 0 \leq i < 3m$. Since $0 < D(f^{3km}(x), f^{3km}(y)) \leq \varepsilon_0 \leq \frac{1}{4}$, by Lemma 3.7, there exists $p \geq 2$ such that $(f^{3km}(x), f^{3km}(y)) \in W_{p-1} - W_p$. Since $f^{3km}(y) \in W_d^s(f^{3km}(x),e) \cap f^{3km}(x)W_{p-1}$,

$$\begin{aligned}
f^n(y) = f^i(f^{3km}(y)) &\in f^i(W_d^s(f^{3km}(x),e) \cap f^{3km}(x)W_{p-1}) \\
&= f^i(W_d^s(f^{3km}(x),e) \cap f^{3km}(x)V_{(p-2)m+1}) \\
&= W_d^s(f^n(x),e) \cap f^n(x)V_{(p-2)m+i+1}.
\end{aligned}$$

Thus $(f^n(x), f^n(y)) \in V_{(p-2)m+i+1} \subset V_{(p-2)m+1} = W_{p-1} \subset B_D(\frac{1}{2^{p-1}})$. Since $(f^{3km}(x), f^{3km}(y)) \notin W_p$ and $B_D(\frac{1}{2^{p+1}}) \subset W_p$,

$$(f^{3km}(x), f^{3km}(y)) \notin B_D\Big(\frac{1}{2^{p+1}}\Big).$$

Thus

$$\begin{aligned}
D(f^n(x), f^n(y)) &\leq \frac{1}{2^{p-1}} = \frac{4}{2^{p+1}} \leq 4D(f^{3km}(x), f^{3km}(y)) \\
&\leq 4\frac{1}{2^k}D(x,y) = 4\lambda^{3km}D(x,y) \\
&\leq \frac{4}{\lambda^i}\lambda^n D(x,y) \\
&\leq 8\lambda^n D(x,y).
\end{aligned}$$

By the same way, we can show that if $y \in W_D^u(x, \varepsilon_0)$, then

$$D(f^n(x), f^n(y)) \leq 8\lambda^{-n} D(x, y) \quad \text{for} \quad n \leq 0.$$

f is expansive with respect to D. Let e be an expansive constant for f. We may assume that $\varepsilon_0 \leq \frac{e}{3}$. There exists $0 < \delta_1 \leq \varepsilon_0$ such that

$$D(x, y) \leq \delta_1 \quad \text{implies} \quad D(f(x), f(y)) \leq \varepsilon_0.$$

Since f has the pseudo orbits tracing property with respect to D, there exists $0 < \delta_0 \leq \delta_1$ such that every δ_0-pseudo orbit is δ_1-traced by some point of X. Let $(x, y) \in B_D(\delta_0)$. Define a δ_0-pseudo orbit $(x_n)_{n \in \mathbb{Z}}$ by

$$x_n = \begin{cases} f^n(x) & \text{for} \quad n \geq 0 \\ f^n(y) & \text{for} \quad n < 0. \end{cases}$$

There exists $z \in X$ such that

$$D(f^n(z), x_n) \leq \delta_1 \quad \text{for all} \quad n \in \mathbb{Z}.$$

It is easy to show that z is unique. Define a map $[\ ,\ ] : B_D(\delta_0) \to X$ by $[x, y] = z$. Since

$$\begin{aligned} &D(f^n(x), f^n([x, y])) = D(x_n, f^n(z)) \leq \delta_1 \leq \varepsilon_0 \quad \text{for all} \quad n \geq 0, \\ &D(f^n(y), f^n([x, y])) = D(x_n, f^n(z)) \leq \delta_1 \leq \varepsilon_0 \quad \text{for all} \quad n < 0, \quad \text{and} \\ &D(f^{-1}(y), f^{-1}([x, y])) \leq \delta_1 \quad \text{implies} \\ &\qquad D(f(f^{-1}(y)), f(f^{-1}([x, y]))) = D(y, [x, y]) \leq \varepsilon_0, \end{aligned}$$

$[x, y] \in W_D^s(x, \varepsilon_0) \cap W_D^u(y, \varepsilon_0)$. Let $z_1, z_2 \in W_D^s(x, \varepsilon_0) \cap W_D^u(y, \varepsilon_0)$. Since

$$\begin{aligned} D(f^n(z_1), f^n(z_2)) &\leq D(f^n(z_1), f^n(x)) + D(f^n(x), f^n(z_2)) \\ &\leq 2\varepsilon_0 \leq e \quad \text{for all} \quad n \geq 0 \end{aligned}$$

and

$$\begin{aligned} D(f^n(z_1), f^n(z_2)) &\leq D(f^n(z_1), f^n(y)) + D(f^n(y), f^n(z_2)) \\ &\leq 2\varepsilon_0 \leq e \quad \text{for all} \quad n \leq 0, \end{aligned}$$

$z_1 = z_2$. Thus $W_D^s(x, \varepsilon_0) \cap W_D^u(y, \varepsilon_0) = \{[x, y]\}$. Let us show that [,] is continuous. Given any $\varepsilon > 0$, by lemma 3.2, there exists $m > 0$ such that $V_{m+1} \subset B_D(\varepsilon)$. We can choose $\delta > 0$ such that if $D(x, y) \leq \delta$, then $D(f^i(x), f^i(y)) \leq \varepsilon_0$ for all $-m \leq i \leq m$. Let $(x_1, y_1), (x_2, y_2) \in B_D(\delta_0)$. $D(x_1, x_2) \leq \delta, D(y_1, y_2) \leq \delta$. Since $D(f^i(x_1), f^i(x_2)) \leq \varepsilon_0$ for $-m \leq i \leq m$,

$$D(f^i(x_1), f^i([x_1, y_1])) \leq \varepsilon_0 \quad \text{for} \quad i \geq 0, \quad \text{and}$$
$$D(f^i(x_2), f^i([x_2, y_2])) \leq \varepsilon_0 \quad \text{for} \quad i \geq 0,$$
$$D(f^i([x_1, y_1]), f^i([x_2, y_2])) \leq 3\varepsilon_0 \leq e \quad \text{for} \quad 0 \leq i \leq m.$$

Since $D(f^i(y_1), f^i(y_2)) \leq \varepsilon_0$ for $-m \leq i \leq m$,

$$D(f^i(y_1), f^i([x_1, y_1])) \leq \varepsilon_0 \quad \text{for} \quad i \leq 0, \quad \text{and}$$
$$D(f^i(y_2), f^i([x_2, y_2])) \leq \varepsilon_0 \quad \text{for} \quad i \leq 0,$$
$$D(f^i([x_1, y_1]), f^i([x_2, y_2])) \leq 3\varepsilon_0 \leq e \quad \text{for} \quad -m \leq i \leq 0.$$

Thus $([x_1, y_1], [x_2, y_2]) \in V_{m+1} \subset B_D(\varepsilon)$. So $D([x_1, y_1], [x_2, y_2]) \leq \varepsilon$. Hence [,] is continuous. Therefore f is hyperbolic with respect to D.

References

1. E. M. Coven and W. L. Reddy, *Positively expansive maps of compact manifolds in Global Theory of Dynamical systems*, Lecture Notes in Mathematics, Springer-Verlag, New York (1980), 96-110.
2. A. H. Frink, *Distance functions and the metrization problem*, Bull. Amer. Math. Soc. **43** (1937), 1408-1411.
3. J. L. Kelley, *General Topology*, Van Nostrand, Princeton, N. J., 1955.
4. R. Mañe, *Ergodic Theory and Ddifferentiable Dynamics*, Springer-Verlag, 1987.
5. J. Ombach, *Consequences of the pseudo orbits tracing property and expansiveness*, J. Austral. Math. Soc. **43** (1987), 301-313.
6. J. S. Park, S. K. Choi, D. P. Chi and K. S. Koo, *Topologically stable homeomorphisms*, Bull. Korean Math. Soc. **28** (1991), 39-49.

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