Bull. Korean Math. Soc. **32** (1995), No. 1, pp. 35–43

CONJUGATE ACTION IN A LEFT ARTINIAN RING

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If R is a left Artinian ring with identity, G is the group of units of R and X is the set of nonzero, nonunits of R, then G acts naturally on X by conjugation. It is shown that if the conjugate action on X by G is trivial, that is, $gx = xg$ for all $g \in G$ and all $x \in X$, then R is a commutative ring. It is also shown that if the conjugate action on X by G is transitive, then R is a local ring and $J^2 = (0)$ where J is the Jacobson radical of R. In addition, if G is a simple group, then R is isomorphic to $Z_2[x]/(x^2+1)$ or Z_4.

1. Introduction and basic definitions

Let R be a ring with identity, let G denote the group of units of R and let X denote the set of nonzero, nonunits of R. We call the action, $\phi : G \times X \to X$ defined by $\phi(g,x) = gxg^{-1}$ for all $g \in G$ and all $x \in X$, the conjugate action or simply conjugation. We define for each $x \in X$, the orbit $0(x) = \{\phi(g,x) : g \in G\}$. We say that the action ϕ is transitive on X if there is an $x \in X$ with $0(x) = X$. Also we say that the action ϕ is trivial if for each $x \in X$, $0(x) = \{x\}$.

An element $a \in R$ is said to be left quasi-regular if there exists $r \in R$ such that $r + a + ra = 0$. In this case the element r is called a left quasi-inverse of a. A (right, left or two-sided) ideal I of R is said to be left quasi-legular if every element of I is left quasi-regular. Similarly, $a \in R$ is said to be right quasi-regular if there exists $r \in R$ such that $a + r + ar = 0$. Right quasi-inverse and right quasi-regular ideals are defined analogously. It is clear that if R has an identity 1, then a is left [resp. right] quasi-regular if and only if $1 + a$ is left [resp. right] invertible. The Jacobson radical of R is defined by the left quasi-regular left ideal which contains every left quasi-regular ideal of R and is denoted by $J(R)$ (or simply J).

Received June 25, 1993.

A ring R is local ring provided that $X \cup \{0\}$ is an ideal of R. In particular, if R is a local ring, then $X \cup \{0\}$ is the unique maximal (right, left or two-sided) ideal of R and $J = X \cup \{0\}$ (See [1], p 170, Proposition 15.15)

A ring R is said to be semisimple if its Jacobson radical J is zero. We note that R/J is semisimple ring.

In [4], Wedderburn-Artin have shown that if R is a semisimple left Artinian ring, then R is isomorphic to a direct sum of a finite number of simple rings. Hence we obtain the following:

THEOREM 1.1. *If R is a left Artinian ring with identity, then $R/J \cong \oplus_{i=1}^{n} M_i\ (D_i)$ where $M_i(D_i)$ is the set of all the $n_i \times n_i$ matrices over a division ring D_i for each $i = 1, 2, \cdots, n$ and for some positive integer n.*

Proof. See [4, Theorem 2.14, p.431 and Theorem 3 3, p.435].

In Section 2, we show that if R is a left Artinian ring with identity such that the conjugate action of G on X is trivial, then R/J is isomorphic to a finite direct sum of fields and R is commutative.

In Section 3, we show that if R is a left Artinian ring with identity such that the conjugate action of G on X is transitive, then R is a local ring and $J^2 = (0)$, in addition, if G is simple, then R is commutative and R is isomorphic to Z_4 or $Z_2[x]/(x^2+1)$.

2. Trivial conjugate action on X

We begin with the following lemma:

LEMMA 2.1. *Let R be a ring, and let G^* be the group of units of R/J. Then $g \in G$ if and only if $g + J \in G^*$.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) Suppose that $g^* = g + J \in G^*$. Then there exists $h^* = h + J \in G^*$ such that $g^*h^* = h^*g^* = 1^*$ where 1^* is the identity of G^*. So $1 - gh$ and $1 - hg \in J$. By the definition of $J, 1 + J \subseteq G$ and so gh and $hg \in G$. It is clear that $g \in G$.

LEMMA 2.2. *Let R be a ring. Then $a \in R$ is left quasi-regular if and only if $a + J \in R/J$ is left quasi-regular.*

Proof. It follows easily from Lemma 2.1.

LEMMA 2.3. *Let R be a left Artinian ring with identity 1. If the conjugate action of G on X is trivial, that is, $gx = xg$ for all $g \in G$ and all $x \in X$, then $R/J \cong \oplus_{i=1}^{n} F_i$ where F_i is a field for each $i = 1, 2, \cdots, n$ and for some positive integer n.*

Proof. By Theorem 1.1, $R/J \cong \oplus_{i=1}^{n} M_i(D_i)$ where $M_i(D_i)$ is the set of all the $n_i \times n_i$ matrices over a division ring D_i for each $i = 1, 2, \cdots, n$ and for some positive integer n. Since $gx = xg$ for all $g \in G$ and all $x \in X$, $g^*x^* = x^*g^*$ for all $g^* \in G^*$ and all $x^* \in X^*$ (= the set of nonzero, nonunits of R/J) – (#). It is clear that if $n_i \geq 2$ for some i, then $M_i(D_i)$ does not satisfy (#). Hence n_i must to be 1 for each i. Next, we will show that D_i is field for each i. If D_i is not field for some i, then there exist a and $b \in D_i$ such that $ab \neq ba$. Let $a^* = (a_1, \cdots, a_i, \cdots, a_n)$ and $b^* = (b_1, \cdots, b_i, \cdots, b_n)$ with $a_i = a \neq 0$, $a_j = 0$ for $j \neq i$ and $b_i = b \neq 0$ and $b_j = 1$. Then $a^* \in X^*$ and $b^* \in G^*$ and $a^*b^* = (0, \cdots, ab, \cdots, 0) \neq (0, \cdots, ba, \cdots, 0) = b^*a^*$, which contradicts to (#). Hence we have the result.

LEMMA 2.4. *Let R be a ring with identity If the conjugate action on X by G is trivial and a and b are quasi-regular elements of R, then $ab = ba$. In particular, J is a commutative ideal of R.*

Proof. Since $1 + J \subseteq G$ and a and $b \in J \subseteq X$, $(1 + a)b = b(1 + a)$ by assumption. Hence $ab = ba$. Since each element of J is quasi-regular, J is a commutative ideal of R.

Let R be a left Artinian ring with identity such that the conjugate action of G on X is trivial. By Lemma 2.2, $R/J \cong \oplus_{i=1}^{n} F_i$ where F_i is field for each $i = 1, 2, \cdots, n$ and for some positive integer n. For simplicity of notation, we can assume that $R/J = \oplus_{i=1}^{n} F_i$. Let $\phi : R \to R/J$ denote the canonical ephimorphism and for each i, let $R_i = \phi^{-1}(\oplus_{i=1}^{n} H_i)$ where $H_j = \{0_j\}$ where 0_j is the additive identity of F_j for $j \neq i$ and $H_i = F_i$. Let $\phi_i = \phi|_{R_i}$. Then Ker $\phi_i = \{a \in R_i : \Pi_i(\phi_i(a)) = 0_i\}$ where Π_i is the projection of $\oplus_{i=1}^{n} F_i$ to F_i. Note that

Ker $\phi_i = J$ for each $i = 1, 2, \cdots, n$ and each R_i is an ideal of R. Let 1_i^* denote the identity of $\phi_i = \oplus_{i=1}^n H_j$, that is, $1_i^* = \oplus_{i=1}^n a_j$ where $a_j = 0_j$ for $j \neq i$ and $a_i = 1_i$ (= the identity of F_i). Observe that $\phi_i^{-1}(\{1_i^*\})$ is contained in the center of R_i if and only if $\phi_i^{-1}(\{-1_i^*\})$ is contained in the center of R_i.

LEMMA 2.5. *Let $\phi : R \to R'$ be a ring epimorphism. If A and B are subsets of R', then $\phi^{-1}(A + B) = \phi^{-1}(A) + \phi^{-1}(B)$.*

Proof. If $x \in \phi^{-1}(A + B)$, then $\phi(x) = a + b \in A + B$. Since ϕ is onto, there exist $a^* \in A$ and $b^* \in B$ such that $\phi(a^*) = a$ and $\phi(b^*) = b$. So $\phi(x) = a + b = \phi(a^*) + \phi(b^*) = \phi(a^* + b^*) \in \phi(\phi^{-1}(A) + \phi^{-1}(B))$. Hence $x \in \phi^{-1}(A) + \phi^{-1}(B)$.

If $x \in \phi^{-1}(A) + \phi^{-1}(B)$, then $x = a^* + b^*$ where $a^* \in \phi^{-1}(A)$ and $b^* \in \phi^{-1}(B)$. So $\phi(x) = (a^* + b^*) = \phi(a^*) + \phi(b^*) \in A + B$. Hence $x \in \phi^{-1}(A + B)$.

LEMMA 2.6. *If R is a left Artinian ring with identity, then $R = R_1 + R_2 + \cdots + R_n$ where $R_i = \phi^{-1}(\oplus_{i=1}^n H_i)$ with $H_j = \{0j\}$ (0_j is additive identity of F_j) for $j \neq i$ and $H_i = F_i$.*

Proof. Let $F_i^* = \oplus_{i=1}^n H_i$ for each i. Then $\oplus_{i=1}^n F_i = F_1^* + F_2^* + \cdots + F_n^*$. Hence $R = \phi^{-1}\phi(R) = \phi^{-1}(R/J) = \phi^{-1}(\oplus_{i=1}^n F_i) = \phi^{-1}(F_1^* + F_2^* + \cdots + F_n^*) = \phi^{-1}(F_1^*) + \phi^{-1}(F_2^*) + \cdots + \phi^{-1}(F_n^*) = R_1 + R_2 + \cdots + R_n$ by Lemma 2.6.

LEMMA 2.7. *Let R be a ring with identity such that the conjugate action on by G, X is trivial and $R/J = \oplus_{i=1}^n F_i$ where each F_i is a field. If $\phi_i^{-1}(\{1_i^*\}) \subseteq Z(R_i)$ (= center of R_i), then R_i is commutative ideal of R for each i.*

Proof. Since R_i is an ideal of R, if $a \in R_i$, then a is quasi-regular in R_i if and only if a is quasi-regular in R. Hence by Lemma 2.2, if $a \in R_i$, then a is quasi-regular in R_i if and only if $\phi(a)$ is quasi-regular in R/J, that is, $\phi_i(a)$ is quasi-regular in $F_i^* = \oplus_{i=1}^n H_j$ where $H_j = \{0_j\}$ for $j \neq i$ and $H_i = F_i$. Hence for $a \in R_i$, a is quasi-regular if $i = 1$ and only if $\Pi_i(\phi_i(a)) + 1_i \neq 0_i$.

Now let $a, b \in R_i$. If a and b are quasi-regular, then $ab = ba$ by Lemma 2.4. If a is not quasi-regular, then $\Pi_i(\phi_i(a)) + 1_i = 0_i$, that is, $a \in \phi_i^{-1}(\{-1_i^*\})$. Thus a is in the center of R_i and so $ab = ba$. Simailarly, if b is not quasi-regular, then $ab = ba$.

LEMMA 2.8. *Let R be a ring with identity such that the conjugate action on X by G is trivial and $R/J = \oplus_{i=1}^n F_i$ where each F_i is a field. If $\phi_i^{-1}(\{1_i^*\}) \subseteq Z(R_i)$ (= center of R_i) for each $i = 1, 2, \cdots, n$, then R is a commutative ring.*

Proof. Let $a \in R_i$ and $b \in R_j$ for $i \neq j$ $(1 \leq i,\ j \leq n)$. By Lemma 2.6, it suffices to show that $ab = ba$. By Lemma 2.4, we may assume that both a and b are not quasi-regular. Without loss of generality, we may assume that a is not quasi-regular. Then $\Pi_i(\phi_i(a)) = -1_i$. Since $ab = ba$ if and only if $(-a)b = b(-a)$, we may assume that $\Pi_i(\phi_i(a)) = 1_i$. Now $ab, ba \in R_i \cap R_j$ since R_i and R_j are ideals of R. But for $i \neq j$, $R_i \cap R_j = J$. So $ab, ba \in J$. Since $J \subseteq Z(R_i)$ for each i, ab and ba are in $Z(R_i)$ for each i. Hence $a(ab) = (ab)a = a(ba) = (ba)a$, that is, $a^2b = ba^2$. Since $\Pi_i(\phi_i(a^2 - a)) = 0_i$, $a^2 - a \in J$. So $(a^2 - a)b = b(a^2 - a)$. Hence $-ab = -ba$, that is, $ab = ba$.

THEOREM 2.9. *Let R be a left Artinian ring with identity. If the conjugation on X by G is trivial, then R is a commutative ring.*

Proof. By Lemma 2.3, we can assume that $R/J = \oplus_{i=1}^n F_i$ where F_i is field for $i = 1, 2, \cdots, n$. By Lemma 2.8, it is enough to show that $\phi_i^{-1}(\{1_i^*\}) \subseteq Z(R_i)$ for each $i = 1, 2, \cdots, n$. Note that $\phi_i^{-1}(\{1_i^*\}) \subseteq Z(R_i)$ if and only if $\phi_i^{-1}(\{-1_i^*\}) \subseteq Z(R_i)$. Let $a \in \phi_i^{-1}(\{-1_i^*\})$ and $b \in R_i$. If $a \in \phi_i^{-1}(\{-1_i^*\})$, then $\phi_i(a) = -1_i^*$ and $\Pi_i(\phi_i(a) = -1_i$. As in the proof of Lemma 2.7, if $a \in R_i$, a is quasi-regular if and only if $\Pi_i(\phi_i(a)) + 1_i \neq 0_i$. So a is not quasi-regular and so $1 + a \in X \cup \{0\}$. If b is quasi-regular, then $1 + b \in G$. So $(1 + a)(1 + b) = (1 + b)(1 + a)$ by assumption, and consequently $ab = ba$. If b is not quasi-regular, then $\Pi_i(\phi_i(a)) = -1_i$. Let $e_i \in R_i$ be such that $\Pi_i(e_i) = -1_i^*$. Then $\Pi_i(\phi_i(a - e_i)) = 0_i$, and so $a - e_i \in \text{Ker } \phi_i = J$.

Similarly, $b - e_i \in J$. Then $a = e_i + x$, $b = e_i + y$ for some x and $y \in J$. Note that $1 + x, 1 + y \in G$ and $1 + e_i \in X \cup \{0\}$ since e_i is not quasi-rgular. So $(1+x)(1+e_i) = ((1+e_i)(1+x)$ and cenquently $xe_i = e_ix$. By the similar argument we have that $ye_i = e_iy$. So $ab = (e_i + x)(e_i + y) = e_i^2 + xe_i + e_iy + xy = e_i^2 + e_ix + ye_i + yx = (e_i + y)(e_i + x) = ba$. Hence

R is commutative.

EXAMPLE 2.10. Let Q be the field of rational numbers. Let

$$R = \{(a_{ij}) \in M_2(Q) : a_{11} = a_{22}, a_{21} = 0\}.$$

Note that $G = \{(a_{ij}) \in R : a_{11} = a_{22} \neq 0\}$, $X = \{(a_{ij}) \in R : a_{11} = a_{22} = 0, a_{12} \neq 0\}$ and the conjugation of G on X is trivial, that is, $gx = xg$ for all $g \in G$ and all $x \in X$, and so R is commutative.

3. Transitive conjugate action in a left Artinian ring

Recall that the action ϕ is said to be transitive on X if there is an $x \in X$ with $0(x) = \{\phi(g, x) : g \in G\} = X$.

LEMMA 3.1. *Let R be a ring with identity. If the conjugate action on X by G is transitive, that is, there is an $x \in X$ with $0(x) = \{gxg^{-1} : g \in G\} = X$, then x is not zero-divisor if and only if y is not zero-divisor for any $y \in X$.*

Proof. If x is not zero-divisor and $ay = ya = 0$ for some $a \in R$, then $a(gxg^{-1}) = (gxg^{-1})a = 0$ for some $g \in G$, and so $(ag)x = x(g^{-1}a) = 0$. As x is not zero-divisor $ag = g^{-1}a = 0$, and so $a = 0$. The similar argument shows that the converse also holds.

LEMMA 3.2. *Let R be the ring of $n \times n$ matrices over a division ring D for any positive integer n. Then every nonzero, nonunit element of R is zero-divisor.*

Proof. Let A be a nonzero, nonunit element of R. If A has r-th row (resp. s-th column) as zero-row, then we can choose $X = (x_{ij}) \in R$ (resp. $Y = (y_{ij}) \in R$) satisfying $x_{rr} \neq 0$ and $x_{ij} = 0$ for i or $j \neq r$ so that $XA = 0$ (resp. $AY = 0$). Hence A is zero-divisor.

Suppose that A has no zero rows (resp. no zero columns). By using the elementary theory in linear algebra, we can obtain an upper-triangular (resp. a low-triangular) martix B (resp. C) from A by means of finite number of elementary row (resp. column) operations and then $S_1 A = B$ (resp. $T_1 A = C$) for some nonsingular matrices S_1 and

$T_1 \in R$. Let b_{ii} (resp. c_{ii}) be the diagonal entries of (resp. C). If b_{ii} (resp. c_{ii}) $\neq 0$ for all i, then by the above argument we can also obtain a diagonal martix D (resp. D') from B (resp. C) by means of finite number of elementary row (resp. column) operations and then $S_2 B = D$ (resp. $T_2 C = D'$) for some nonsingular matices S_2 and $T_2 \in R$. Thus $S_2 S_1 A = D$ (resp. $T_2 T_1 A = D'$) and then A is non-singular, a contradiction. Hence $b_{ii} = 0$ (resp. $c_{ii} = 0$) for some i. Let r (resp. s) be the largest integer so that $b_{ii} = 0$ (resp. $c_{ii} = 0$). Then B (resp. C) has r-th row (resp. s-th column) as zero. So $XB = XS_1 A = 0$ (resp. $BY = AT_1 Y = 0$) for some nonzero singular matrices $X, Y \in R$. If $XS_1 = T_1 Y = 0$, then $X = Y = 0$, a contradiction. Consequently, A is zero-divisor.

COROLLARY 3.3. *Let R be a semisimple left Artinian ring. Then every nonzero, nonunit of R is zero-divisor.*

Proof. It follows from Theorem 1.1 and Lemma 3.2.

THEOREM 3.4. *Let R be a left Artinian ring with identity. If the conjugate action on X by G is transitive, then every $x \in X$ is zero-divisor.*

Proof. By assumption, there is an $x \in X$ with $0(x) = \{gxg^{-1} : g \in G\} = X$. By Lemma 3.1, it is enough to show that x is zero-divisor. Assume that x is not zero-divisor. If $x \in J$, then $x^n = 0$ for some positive integer n as J is nilpotent ideal of R. Hence if x is not a zero-divisor, then $x = 0$, a contradiction. Suppose that $x \in R \setminus J$. Let $x^* = x + J \in R/J$. By Lemma 2.1, x^* is not unit of R/J. Hence by Corollary 3.3, x^* is zero-divisor and so $x^* y^* = z^* x^* = 0^* = J$ for some nonzero $y^*, z^* \neq J$ and then $xy, zx \in J$. Thus $(xy)^s = 0$ (resp. $(zx)^t = 0$) for some positive integers s, t. By the above argument, $xy = zx = 0$. If x is not zero-divisor, then $y = z = 0$, a contradiction. Hence x is a zero-divisor.

THEOREM 3.5. *Let R be a left Artinian ring with identity. If the conjugate action on X by G is transitive, then R is a local ring and $J^2 = (0)$.*

Proof. By assumption, there is an $x \in X$ with $0(x) = \{gxg^{-1} : g \in G\} = X$. First, we will show that $XX \subseteq X \cup \{0\}$. Assume that there exist $y, z \in X$ such that $yz \notin X \cup \{0\}$ and then $yz \in G$. Since y is zero-divisor by Theorem 3.4, $ay = 0$ for some $a \in X$, and so $(ay)z = a(yz) = 0$. Since $yz \in G, a = 0$, a contradiction. Clearly, $GX, XG \subseteq X \cup \{0\}$, and hence $RX, XR \subseteq X \cup \{0\}$. Next, we will show that $X \cup \{0\}$ is closed under addition. Since x is zero-divisor, by the above argument there exists $s \in X$ such that $xs = 0$. Since the conjugate action of g on X is transitive, there exists $g \in G$ such that $s = \mathrm{gxg}^{-1}$. Then $0 = xs = x(\mathrm{gxg}^{-1})$ and hence $(gx)(gx) = 0$. Let $gx = x_0$. Then $x_0^2 = 0$. Now if t is any element of X, then there exists $h \in G$ such that $t = hx_0h^{-1}$. So $t^2 = hx_0^2h^{-1} = 0$. Conquently for any $t_1, t_2 \in X, (t_1 + t_2)^2 = t_1^2 + t_1t_2 + t_2t_1 + t_2^2 = t_1t_2 + t_2t_1$, and so $(t_1 + t_2)^4 = (t_1t_2 + t_2t_1)^2 = 0$. Hence $X \cup \{0\}$ is closed under addition. Thus R is a local ring and $J = X \cup \{0\}$.

Finallly, we will show that $J^2 = (0)$. Let a and b be arbitrary elements of J. Then there exist g and h of G such that $a = gx_0g^{-1}$ and $b = hx_0h^{-1}$. If $ab \neq 0$, then $ab = (gx_0g^{-1})(hx_0h^{-1}) \neq 0$, and then $x_0g^{-1}hx_0 \neq 0$. S0 $(g^{-1}hx_0)(g^{-1}hx_0) = (g^{-1}hx_0)^2 \neq 0$ a contradiction. Thus $J^2 = (0)$.

EXAMPLE 3.6. Let $F = Z_2[x]/ < x^2 + x + 1 >$ be a field of order 4 and let $R = \{(a_{ij}) \in M_2(F) : a_{11} = a, a_{22} = a^2, a_{21} = 0\}$.

Note that R is a ring under the addition mod 2 and multiplication mod 2 of matrices, R is not commutative, $G = \{(a_{ij}) \in R : a_{11}, a_{22} \neq 0\}$, $X = \{(a_{ij}) \in R : a_{11} = a_{22} = 0,\ a_{12} \neq 0\}$ and the conjugation on X by G is transitive. Moreover, R is a local ring and $J^2 = (0)$.

COROLLARY 3.7. *Let R be a left Artinian ring with identity such that the conjugate action of G on X is transitive. If G is simple group, then R is commutative ring and R is isomorphic to $Z_2[x]/(x^2 + 1)$ or Z_4.*

Proof. If the conjugate action of G on X is transitive, then by Theorem 3.5, $J^2 = (0)$. Let $g \in 1 + J$ and $x \in J$ be arbitrary. If $J^2 = (0)$, then $gx = xg$. Note that $1 + J$ is a proper normal subgroup of G. Hence if G is simple, then $1 + J = G$, and so the conjugate action of G on X is trivial. Thus by Lemma 2.9, R is a commutative ring. If R is commutative, then $J = \{0, x\}$.

We note that if $J^2 = (0)$, then a function $\phi : R/J \times J \to J$ defined by $\phi(a + J,\ j) = aj$ for all $a + J \in R/J,\ j \in J$ is well-defined and J is 1-dimensional left vector space over a division ring R/J by Lemma 2.1. Hence $|R/J| = |J| = 2$ and so $|R| = |R/J| \cdot |J| = 4$. By [2] (Theorem 2.5 and its Corollary), R is isomorphic to $Z_2[x]/(x^2 + 1)$ or Z_4.

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