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HOLOMORPHIC SECTIONAL CURVATURE OF THE TANGENT BUNDLE*

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1. Introdution

In order to investigate the differential structure of a Riemannian manifold (M, g), it seems a powerful tool to study the differential structure of its tangent bundle TM. In this point of view, K. Aso [1] studied, using the Sasaki metric $\tilde{g}$, the relation between the curvature tensor on (M, g) and that on $(TM, \tilde{g})$.

On the other hand T. Nagano [2], S. Tachibana and M. Okumura [5] showed that $(TM, \tilde{g})$ has an almost complex structure ϕ, and furthermore this structure ϕ is the almost Hermitian structure with respect to $\tilde{g}$. By virtue of these results, it is natural to consider the holomorphic sectional curvature on $(TM, \tilde{g})$, and to study the relation between the holomorphic sectional curvature of $(TM, \tilde{g})$ and that of (M, g). In this paper, we study the above relation, and we obtain the following theorem as an improvement of K. Also's result [1].

THEOREM. *If the holomorphic sectional curvature of $(TM, \tilde{g})$ is bounded, then (M, g) is flat.*

Manifolds, geometric objects and mappings we discuss in this paper will be assumed to be differentiable and of class C^∞. Throughout this paper, the indices h, i, j run over the range $\{1, 2, \cdots, n\}$ and the indices λ, κ, μ over the range $\{1, 2, \cdots, 2n\}$, and the summation convention is used with respect to those systems of indices.

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2. Preliminaries

Let (M, g) be an n-dimensional connected Riemannian manifold and $(TM, \tilde{g})$ be its tangent bundle, where $\tilde{g}$ denotes the Sasaki metric. Let ∇ (resp. $\tilde{\nabla}$) be the Levi-Civita connection on (M, g) (resp. $(TM, \tilde{g})$) and R (resp. $\tilde{R}$) the curvature tensor of ∇ (resp. $\tilde{\nabla}$).

We first of all recall the almost complex structure ϕ which is due to T.Nagano [2].

Let $\{U;\ x^i\}$ be local coordinates of a point $p \in M$, then a tangent vector ξ at p, which is an element of TM, is expressible in the form (x^i, x^{n+i}), where x^{n+i} are components of ξ with respect to the natural frame $\partial_i = \frac{\partial}{\partial x^i}$. We may consider (x^i, x^{n+i}) as local coordinates of TM.

Put $\Gamma_i{}^h = \{_i{}^h{}_j\} x^{n+j}$, where $\{_i{}^h{}_j\}$ denotes the Christoffel symbols formed by the Riemannian metric g_{ij}. If we define $\phi_\lambda{}^\kappa$ with respect to each local coordinates (x^i, x^{n+i}) of TM by

$$
\begin{aligned}
\phi_i{}^h &= \Gamma_i{}^h \quad & \phi_i{}^{n+h} &= -\delta_i^h - \Gamma_i{}^j\Gamma_j{}^h, \\
\phi_{n+i}{}^h &= \delta_i{}^h, \quad & \phi_{n+i}{}^{n+h} &= -\Gamma_i{}^h,
\end{aligned}
\tag{2.1}
$$

then we can see that $\phi_\lambda{}^\kappa \phi_\kappa{}^\mu = -\delta_\lambda^\mu$ holds. Hence, ϕ defines an almost comlex structure on TM.

From now on we denote by M_p the tangent space at $p \in M$. Let $\pi : TM \longrightarrow M$ be the tangent bundle of M and $TM_{(p,\xi)}$ the tangent space at $(p, \xi) \in TM$. We write simply ξ instead of (p, ξ).

Let $\xi \in TM$ with $\pi\xi = p$. We define the connection map $K : TM_\xi \longrightarrow M_p$ as follows:

For any $X \in TM_\xi$, let $\xi(t) : (-\varepsilon, \varepsilon) \longrightarrow TM$ be a curve in TM such that $\xi'(0) = X$. Then $\xi(t)$ can be regarded as the vector field along the curve $\sigma(t) = \pi \circ \xi(t)$ in M. We define $KX = (\nabla_{\dot{\sigma}})\xi(0)$, where ∇ be the Levi-Civita connection on M.

Let $d\pi : TTM \longrightarrow TM$ be the differential of the projection π. Then the kernel H_ξ of K and the kernel V_ξ of $d\pi$ are both n-dimensional disjoint subspaces of TM_ξ, which are called the horizontal and vertical subspaces at ξ, respectively. For any vector $v \in M_p$ and $\xi \in \pi^{-1}(p)$, there exists a unique vector $v^* \in H_\xi$ (resp. ${}^*v \in V_\xi$) such that $d\pi v^* = v$

(resp. $K\ {}^*v = v$), which will be called the horizontal (resp. vertical) lift of v to ξ. We define the metric $\tilde{g}$ on TM by

$$\tilde{g}(X,Y)(\xi) = g(d\pi X,\ d\pi Y)(p) + g(KX, KY)(p) \tag{2.2}$$

for $X, Y \in TM_\xi$ and $\pi\xi = p$. This metic $\tilde{g}$ is called *Sasaki metric* on TM, which together with ϕ defined in (2.1) determines an almost Hermitian structure on TM [5]. A vector field on TM is called an ***associated vector field with*** $d\pi$ ***and*** K if $d\pi X(\xi_1) = d\pi X(\xi_2)$ and $KX(\xi_1) = KX(\xi_2)$, whenever ξ_1 and ξ_2 are points in the same fiber.

The space of associated vector fields with $d\pi$ and K on TM is denoted by $\mathcal{X}_*(TM)$. Hereafter we put $X_\pi := d\pi X$ and $X_K := KX$ for the vector field $X \in \mathcal{X}_*(TM)$.

3. The curvature and holomorphic sectional curvature on TM

Let $\xi(t) : (-\varepsilon, \varepsilon) \longrightarrow TM$ be a curve in TM with $\xi(0) = \xi$ and $\sigma(t) = \pi \circ \xi(t)$ with $p = \sigma(0)$. Let $X(t)$ be a vector field along $\xi(t)$ in TM, and let $\{\pi^{-1}(U); (x^1, \cdots, x^n, x^{n+1}, \cdots, x^{2n})\}$ a coordinate neighborhood system of TM, where $\{U\ ; (x^1, \cdots, x^n)\}$ is a coordinate neighborhood system of M. Put $\xi(t) = (x^1(t), \cdots, x^n(t), x^{n+1}(t), \cdots, x^{2n}(t))$ and $X(t) = X^i(t)\frac{\partial}{\partial x^i} + X^{n+i}(t)\frac{\partial}{\partial x^{n+i}}$.

By the relation between Christoffel symbols on TM and those on M (cf. [1], [3]), it follows that

$$\begin{aligned} d\pi(\tilde{\nabla} X) &= \nabla(X_\pi) + \frac{1}{2}R(\nabla\xi, \xi)X_\pi + \frac{1}{2}R(X_K,\ \xi)\sigma', \\ K(\tilde{\nabla} X) &= \nabla(X_K) + \frac{1}{2}R(\sigma', X_\pi)\xi, \end{aligned} \tag{3.1}$$

where the parameter t is omitted.

Let $\xi(t) : (-\varepsilon,\ \varepsilon) \longrightarrow TM$ be an integral curve of $X \in \mathcal{X}_*(TM)$. Then, from (2.2), we have

$$\begin{aligned} &d\pi(\tilde{\nabla}_X Y)(\xi) \\ &= \nabla_{X_\pi} Y_\pi(p) + \frac{1}{2}R(X_K,\ \xi)Y_\pi(p) + \frac{1}{2}R(Y_K,\ \xi)X_\pi(p), \\ &K(\tilde{\nabla}_X Y)(\xi) = \nabla_{X_\pi} Y_K(p)\frac{1}{2}R(X_\pi,\ Y_\pi)\xi(p) \end{aligned} \tag{3.2}$$

for any $Y \in \mathcal{X}_*(TM)$(cf. [1]). Since $[X,Y] = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X$, it is clear that

$$(3.3)\quad \begin{aligned} &d\pi[X,Y](\xi) = [X_\pi, Y_\pi](p),\\ &K[X,Y](\xi) = \nabla_{X_\pi} Y_K(p) - \nabla_{Y_\pi} X_K(p) + R(X_\pi, Y_\pi)\xi(p) \end{aligned}$$

(cf. [1]). The restriction of the vector field $\tilde{\nabla}_Y Z$ to the image of $\xi(t)$ is also denoted by the same notation. Then we have

(3.4)

$$\begin{aligned} &d\pi(\tilde{\nabla}_X(\tilde{\nabla}_Y Z))\\ &= \nabla_{X_\pi}\nabla_{Y_\pi} Z_\pi + \frac{1}{2}(\nabla_{X_\pi} R)(Y_K, \xi)Z_\pi + \frac{1}{2}R(\nabla_{X_\pi} Y_K, \xi)Z_\pi\\ &+ \frac{1}{2}R(Y_K, X_K)Z_\pi + \frac{1}{2}R(Y_K,\xi)\nabla_{X_\pi} Z_\pi + \frac{1}{2}(\nabla_{X_\pi} R)(Z_K,\xi)Y_\pi\\ &+ \frac{1}{2}R(\nabla_{X\pi} Z_K, \xi)Y_\pi + \frac{1}{2}R(Z_K, X_K)Y_\pi + \frac{1}{2}R(Z_K,\xi)\nabla_{X_\pi} Y_\pi\\ &+ \frac{1}{2}R(X_K,\xi)\nabla_{Y_\pi} Z_\pi + \frac{1}{4}R(X_K,\xi)R(Y_K,\xi)Z_\pi\\ &+ \frac{1}{4}R(X_K,\xi)R(Z_K,\xi)X_\pi + \frac{1}{2}R(\nabla_{Y_\pi} Z_K,\xi)X_\pi\\ &+ \frac{1}{4}R(R(Y_\pi, Z_\pi)\xi,\xi)X_\pi,\\ &K(\tilde{\nabla}_X(\tilde{\nabla}_Y Z))\\ &= \nabla_{X_\pi}\nabla_{Y_\pi} Z_K + \frac{1}{2}(\nabla_{X_\pi} R)(Y_\pi, Z_\pi)\xi + \frac{1}{2}R(\nabla_{X_\pi} Y_\pi, Z_\pi)\xi\\ &+ \frac{1}{2}R(Y_\pi, \nabla_{X_\pi} Z_\pi)\xi + \frac{1}{2}R(Y_\pi, Z_\pi)X_K + \frac{1}{2}R(X_\pi, \nabla_{Y_\pi} Z_\pi)\xi\\ &+ \frac{1}{4}R(X_\pi, R(Y_K,\xi)Z_\pi)\xi + \frac{1}{4}. \end{aligned}$$

By using the equations above, K. Aso proved the following theorems;

THEOREM A. *$(TM,\tilde{g})$ is flat if and only if (M,g) is flat.*

THEOREM B. *If the sectional curvature of $(TM,\tilde{g})$ is bounded, then $(TM,\tilde{g})$ is flat.*

We first prepare the following lemma to prove the main theorem.

LEMMA. *For any* $X \in H_\xi$, *(resp.* $X \in V_\xi$*),* $\phi X \in V_\xi$ *(resp.* $\phi X \in H_\xi$*). In particular,* $d\pi(\phi X) = X_K$ *and* $K(\phi X) = -X_\pi$*, which imply*

$$d\pi(\tilde{R}(X,\phi X)X) = \begin{cases} -\frac{1}{2}(\nabla_{X_\pi} R)(X_\pi,\xi)X_\pi & \text{if } X \in H_\xi, \\ -\frac{1}{4}R(X_k,\xi)R(X_K,\xi)X_K & \text{if } X \in V_\xi, \end{cases}$$

$$K(\tilde{R}(X,\phi X)X) = \begin{cases} -\frac{1}{4}R(X_\pi,R(X_\pi,\xi)X_\pi)\xi & \text{if } X \in H_\xi, \\ 0 & \text{if } X \in V_\xi. \end{cases}$$

Proof. Let $X = X^i\frac{\partial}{\partial x^i} + X^{n+i}\frac{\partial}{\partial x^{n+i}}$ be a vector field in $\mathcal{X}_*(\mathrm{TM})$. Then

$$\begin{aligned}\phi X &= X^i\phi_i{}^\kappa\frac{\partial}{\partial x^\kappa} + X^{n+i}\phi_{n+i}{}^\kappa\frac{\partial}{\partial x^\kappa} \\ &= (X^i\Gamma_i{}^j + X^{n+j})\frac{\partial}{\partial x^j} + (-X^j + X^i\Gamma_i{}^r\Gamma_r{}^j + X^{n+i}\Gamma_i{}^j)\frac{\partial}{\partial x^{n+j}}.\end{aligned}$$

It follows that for ${}^\forall X \in H_\xi$,

$$d\pi(\phi X) = (X^{n+j} + \{{}_i{}^j{}_k\}x^{n+k}X^i)\frac{\partial}{\partial x^j} = X_K = 0.$$

Similarily, for ${}^\forall X \in V_\xi$,

$$K(\phi X) = -X^j\frac{\partial}{\partial x^j} = -X_\pi = 0.$$

The last two equations follow directly from (3.3). The proof is completed.

By using the lemma above, we can prove the main theorem stated in section 1 as follows:

Proof of main theorem. If $X \in H_\xi$ for $\xi \in TM$ with $\pi\xi = p$, then by Lemma the holomorphic sectional curvature is given by

$$\begin{aligned}G(X) &= \tilde{g}(\tilde{R}(X,\phi X)X,\phi X)(\xi) \\ &= \langle d\pi\tilde{R}(X,\phi X)X, d\pi(\phi X)\rangle(p) + \langle K\tilde{R}(X,\phi X)X, K(\phi X)\rangle(p) \\ &= \frac{1}{4}\langle R(X_\pi,\xi)X_\pi, R(X_\pi,\xi)X_\pi\rangle(p) \\ &= \frac{1}{4} \parallel R(X_\pi,\xi)X_\pi \parallel_p^2,\end{aligned}$$

where $\|\quad\|$ denotes the norm given by g. If $X \in V_\xi$, then

$$G(X) = \frac{1}{4} \| R(X_K, \xi)X_K \|_p{}^2$$

is similarly computed.

Assume that $(TM, \tilde{g})$ is not flat. Since M is not flat by Theorem A, there exist a point $p \in M$ and a vector $v \in M_p$ such that $R(v,\xi)v \neq 0$ for a unit vector ξ orthogonal to v. Since

$$G(v^*) = \frac{1}{4} \| R(v,\xi)v \|^2,$$

and the set of ξ's satisfying such condition is unbounded, $G(v^*)$ is unbounded. By the same way we can vertify that $G({}^*v)$ is unbounded. This is a contradiction, and consequently $(TM, \tilde{g})$ is flat. Thus we complete the proof of our main theorem.

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