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HOMOCLINIC ORBITS FOR HAMILTONIAN SYSTEMS

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0. Introduction

Let $p, q \in \mathbb{R}^n$ and $H : \mathbb{R}^{2n} \to \mathbb{R}^n$ be differentiable. An autonomous Hamiltonian system has the form

$$\dot{p} = -\frac{\partial H}{\partial q}(p,q), \quad \dot{q} = \frac{\partial H}{\partial p}(p,q). \tag{0.1}$$

When

$$H(p,q,t) = 1/2|p|^2 - 1/2\langle L(t)q, q\rangle + V(t,q)$$

with $L(t)$ an $n \times n$ symmetric matrix, the equation (0.1) becomes to

$$\begin{cases} \dot{q} = p, \\ \dot{p} = L(t)q - V_q(t,q). \end{cases} \tag{0.2}$$

Thus

$$\ddot{q} - L(t)q + V_q(t,q) = 0. \tag{HS}$$

Let $E := W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ under the usual norm

$$\|q\|^2 := \int_{-\infty}^{\infty} (|\dot{q}|^2 + |q|^2)\, dt, \quad q \in E.$$

Thus E is a Hilbert space and $E \subset C^0(\mathbb{R}, \mathbb{R}^n)$, the space of continuous function q on $\mathbb{R}$ such that $q(t) \to 0$ as $|t| \to \infty$. Now let

$$I(q) = \frac{1}{2}\int_{-\infty}^{\infty} (|\dot{q}|^2 + \langle q, L(t)q\rangle)\, dt - \int_{-\infty}^{\infty} V(t,q)\, dt$$

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be the corresponding functional associated with (HS). We assume that $L(t)$ is T-periodic in t, is symmetric and positive definite uniformly on $[0,T]$. Then

$$\|q\|^2 := \int_{-\infty}^{\infty} (|\dot{q}|^2 + \langle q, L(t)q\rangle)\,dt$$

can and will be taken as an equivalent norm on E. Note that

$$I(\tau_j q) = I(q),$$

where $\tau_j q(t) := q(t - jT)$. Hence I posseses a $\mathbb{Z}$-action.

V.Coti Zelati and P. Rabinowitz[**3**] studied the existence of infinitely many homoclinic solutions of the Hamiltonian system of ordinary differential equations:

(HS) $\ddot{q} - L(t)q + V_q(t,q)$

assuming that L and V satisfy:

(L) For each $t \in \mathbb{R}$, $L(t)$ is a symmetric positive definite $n \times n$ matrix and is continuous and T-periodic in t,

(V_1) $V \in C^2(\mathbb{R} \times \mathbb{R}^n)$ and is T-periodic in t,

(V_2) $V_{qq}(t,0) = 0$,

(V_3) there is a $\mu > 2$ such that

$$0 < \mu V(t,q) \le \langle q, V_q(t,q)\rangle$$

for all $t \in \mathbb{R}$ and $q \in \mathbb{R}^n \setminus \{0\}$.

($*$) there is an $\alpha > 0$ such that $I^{c+\alpha}/\mathbb{Z}$ contrains only finitely many critical points of I.

Moreover they have suggested that the condition ($*$) could be replaced with a weaker condition if we further require that V satisfy the following condition

$$(V_4)\qquad \text{For all } \ \xi \in S^{n-1},\ s \to 1/s\,\langle \xi, V_q(t, s\xi)\rangle \ \text{ is an increasing function of } s.$$

In this work we give a condition on V and this will replace the crucial condition ($*$).

1. Preliminaries

Let's use the following notations;

$$I_a := \{q \in E \mid I(q) \geq a\}, \quad I^b := \{q \in E \mid I(q) \leq b\},$$
$$I_a^b := I_a \cap I^b, \quad \mathcal{K} := \text{the set of critical points of } I,$$
$$\mathcal{K}_a^b := \mathcal{K} \cap I_a^b.$$

Let X be a Banach space.

DEFINITION 1.1. $\phi \in \mathcal{C}^1(X, \mathbb{R})$ satisfies the *Palais-Smale condition* (PS) if every sequence (u_j) in X such that $(\phi(u_j))$ is bounded and

$$\phi'(u_j) \to 0 \quad \text{for} \quad j \to \infty$$

contains a convergent subsequence.

The Palais-Smale condition is a compactness condition on ϕ which replaces the compactness of the manifold in the classical Lusternik-Schnirelman theory. We will seek solutions of (HS) as a critical points of the functional I associated with (HS). Note that the key roles (PS) plays in the proof of the standard deformation theorem is that it provides a $\delta > 0$ such that $\|I'(x)\| \geq \delta$ for all $x \in I_{b-\epsilon}^{b+\epsilon}$ for some $\epsilon > 0$ if $\mathcal{K}(b) := \mathcal{K}_b^b = \emptyset$ and appropriately modified statement if $\mathcal{K}(b) \neq \emptyset$. But our functional I does not satisfy the (PS) condition. However we can overcome this difficulty in the following way. From now on we assume that V safisfies (V_1)-(V_4) and that L satisfies the condition (L).

Given $q \in E \setminus \{0\}$, define a function $f : (0, \infty) \to \mathbf{R}$ by

$$\begin{aligned} f(s) &= I(sq) \\ &= \frac{s^2}{2} \int_{-\infty}^{\infty} (|\dot{q}|^2 + \langle q, L(t)q\rangle)dt - \int_{-\infty}^{\infty} V(t, sq)dt. \end{aligned}$$

Then

$$\begin{aligned} f'(s) &= s \int_{-\infty}^{\infty} (|\dot{q}|^2 + \langle q, L(t)q\rangle)dt - \int_{-\infty}^{\infty} \langle q, V_q(t, sq)\rangle dt \\ &= s\left(\int_{-\infty}^{\infty} (|\dot{q}|^2 + \langle q, L(t)q\rangle)dt - \frac{1}{s}\int_{-\infty}^{\infty} \langle q, V_q(t, sq)\rangle dt\right). \end{aligned}$$

Now (V_4) implies that $f : (0,\infty) \to \mathbf{R}$ has a unique maximum point. Moreover (V_1)-(V_3) implies that

$$V(t,x)\begin{cases} \le M|x|^\mu & \text{uniformly in } t \text{ for} \quad |x| \le 1, \\ \ge m|x|^\mu & \text{uniformly in } t \text{ for} \quad |x| \ge 1. \end{cases}$$

Here

$$\begin{aligned} m &= \min_{\substack{t\in\mathbf{R} \\ |x|=1}} V(t,x) > 0 \quad \text{and} \\ M &= \max_{\substack{t\in\mathbf{R} \\ |x|=1}} V(t,x) > 0. \end{aligned}$$

Hence $f(s) \to -\infty$ as $s \to +\infty$. Observe also that $I(q) = \frac{1}{2}\|q\|^2 + o(\|q\|^2)$. Therefore 0 is an isolated singular point of I. Choose a point $e \ne 0$ such that $I(e) \le 0$. Let

$$c = \inf_{g\in\Gamma_e} \max_{\theta\in[0,1]} I(g(\theta)),$$

where

$$\Gamma_e = \{g \in C([0,1],E) : g(0) = 0,\ g(1) = e\}.$$

Since $I(q) = \frac{1}{2}\|q\|^2 + o(\|q\|^2)$, $\quad c > 0$. Usually the value of c depends on the choice of e. But we have the following

LEMMA 1.1. *If V satisfies (V_1)-(V_3), then c is independent of the choice of e.*

Proof. Define a function $f : (0,\infty) \to \mathbf{R}$ by

$$\begin{aligned} f(s) &= I(sq) \\ &= \frac{s^2}{2}\int_{-\infty}^{\infty} (|\dot q|^2 + \langle q, L(t)q\rangle)dt - \int_{-\infty}^{\infty} V(t,sq)dt. \end{aligned}$$

Then

$$\begin{aligned} f'(s) &= s\int_{-\infty}^{\infty} (|\dot q|^2 + \langle q, L(t)q\rangle)dt - \int_{-\infty}^{\infty} \langle q, V_q(t,sq)\rangle dt \\ &\le s\int_{-\infty}^{\infty} (|\dot q|^2 + \langle q, L(t)q\rangle)dt - \frac{\mu}{s}\int_{-\infty}^{\infty} V(t,sq)dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{\mu}{s}\Big(\frac{s^2}{\mu}\int_{-\infty}^{\infty}(|\dot q|^2 + \langle q, L(t)q\rangle)dt - \int_{-\infty}^{\infty} V(t, sq)dt\Big)\\
&\le \frac{\mu}{s}\Big(\frac{s^2}{2}\int_{-\infty}^{\infty}(|\dot q|^2 + \langle q, L(t)q\rangle)dt - \int_{-\infty}^{\infty} V(t, sq)dt\Big)\\
&= \frac{\mu}{s} f(s).
\end{aligned}
$$

Hence we obtain $f'(s) - \mu/sf(s) \le 0$. This implies that $f(s)/s^{\mu}$ is a decreasing funciton of s. Therefore any two points $e_1 \ne 0$ and $e_2 \ne 0$ such that $e_1 \in I^0$ and $e_2 \in I^0$ can be joined by a path lying in I^0. This proves that c is independent of the choice e. □

To define an another intrinsic constant $\overline{c}$,we need the following

LEMMA 1.2. *If* $q \in \mathcal{K}$, *then* $I(q) \ge (\frac{1}{2} - \frac{1}{\mu})\|q\|^2$.

Proof.

$$
\begin{aligned}
I(q) &= \frac{1}{2}\int_{-\infty}^{\infty}(|\dot q|^2 + \langle q, L(t)q\rangle)dt - \int_{-\infty}^{\infty} V(t, q)dt\\
\langle I'(q), q\rangle &= \int_{-\infty}^{\infty}(|\dot q|^2 + \langle q, L(t)q\rangle)dt - \int_{-\infty}^{\infty} \langle q, V_q(t, q)\rangle dt\\
&= 0.
\end{aligned}
$$

Hence

$$
\begin{aligned}
I(q) &= I(q) - \frac{1}{2}\langle I'(q), q\rangle = \int_{-\infty}^{\infty}\Big(\frac{1}{2}\langle q, V_q(t, q)\rangle - V(t, q)\Big)dt\\
&\ge \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\int_{-\infty}^{\infty}\langle q, V_q(t, q)\rangle dt = \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\int_{-\infty}^{\infty}(|\dot q|^2 + \langle q, L(t)q\rangle)dt\\
&= \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|q\|^2. \qquad \square
\end{aligned}
$$

Let

$$
\overline{c} = \inf_{q \in \mathcal{K}\setminus\{0\}} I(q).
$$

Since 0 is an isolated singular point, Lemma 1.2 implies that $\overline{c} > 0$. We now have two inherently defined constants c and $\overline{c}$. To compare the two numbers c and $\overline{c}$,we need the following two Lemmas.

LEMMA 1.3. ([4]) *Let K be a compact metric space, $K_0 \subset K$ a closed set, X a Banach space, $\chi \in C(K_0, X)$ and let us define a complete metric space*

$$M = \{g \in C(K, X); g(s) = \chi(s) \text{ if } s \in K_0\}$$

with the usual distance d. Let $\varphi \in C^1(X, \mathbf{R})$ and let us define

$$c = \inf_{g\in M} \max_{s\in K} \varphi(g(s)).$$

Then for each sequence (f_k) in M such that

$$\max_K \varphi(f_k) \to c,$$

there exists a sequence (v_k) in X such that

$$\begin{aligned} &\varphi(v_k) \to c, \\ &dist(v_k, f_k(K)) \to 0, \\ &|\varphi'(v_k)| \to 0 \quad \text{as} \quad k \to +\infty. \end{aligned}$$

LEMMA 1.4. ([3]) *Let $(u_m) \subset E$ be such that $I(u_m) \to b > 0$ and $I'(u_m) \to 0$. Then there is an $\ell \in \mathbf{N}$ with ℓ bounded above by a constant depending only on b, normalized functions $v_1, v_2, \cdots, v_\ell \in \mathcal{K} \setminus \{0\}$, a subsequence of (u_m), and corresponding $(k_m^i) \subset \mathbf{Z}$, $1 \leq i \leq \ell$, such that*

$$\|u_m - \sum_1^\ell \tau_{k_m^i} v_i\| \to 0, \qquad \sum_1^\ell I(v_i) = b,$$

and, for $i \neq j$,

$$|k_m^i - k_m^j| \to +\infty$$

as $m \to \infty$ along the subsequence.

In the above we say that a function v is normalized if

$$\|v\|_{L^\infty} = \max_{t\in\mathbf{R}} |v(t)|$$

occurs for $t \in [0, T]$ and $|v(t)| < \|v\|_{L^\infty}$ for $t < 0$. We are now ready to show that $c = \overline{c}$.

THEOREM 1.1. *If V satisfies the conditions (V_1)-(V_4), then $c = \overline{c}$.*

Proof. Suppose $c < \overline{c}$. By Lemma 1.3 there exists a sequence $(u_m) \subset E$ such that $I(u_m) \to c$ and $I'(u_m) \to 0$. Since $c > 0$, we can apply Lemma 1.4 to obtain a normalized critical points $v_1, v_2, \cdots, v_\ell$ such that

$$\sum_{i=1}^{\ell} I(v_i) = c.$$

But this contradicts the fact that $\overline{c} = \inf_{q\in\mathcal{K}\setminus\{0\}} I(q)$. Therefore $c \geq \overline{c}$. On the other hand, given any $q \in \mathcal{K} \setminus \{0\}$, consider

$$\begin{aligned} f(s) &= I(sq) \\ &= \frac{s^2}{2}\int_{-\infty}^{\infty} (|\dot{q}|^2 + \langle q, L(t)q\rangle)dt - \int_{-\infty}^{\infty} V(t, sq)dt. \end{aligned}$$

Observe that

$$f'(s) = s\left(\int_{-\infty}^{\infty} (|\dot{q}|^2 + \langle q, L(t)q\rangle)dt - \frac{1}{s}\int_{-\infty}^{\infty} \langle q, V_q(t, sq)\rangle dt\right).$$

Since $q \in \mathcal{K} \setminus \{0\}$, $f'(1) = 0$. Now (V_4) implies that f attains its maximum value at $s = 1$. Therefore $c \leq f(1) = I(q)$ for any $q \in \mathcal{K}\setminus\{0\}$. Hence $c \leq \overline{c}$. □

Remember that I does not satisfy the (PS) condition. However we can show that c is a critical value of I.

THEOREM 1.2. *If V satisfies the conditions (V_1)-(V_4), then c is a critical value of I.*

Proof. Choose a sequence $(q_m) \subset \mathcal{K} \setminus \{0\}$ such that $I(q_m) \to \overline{c} = c$. Since $I(q) \geq (\frac{1}{2} - \frac{1}{\mu})\|q\|^2$ for all $q \in \mathcal{K}$, (q_m) is bounded in E. Hence there exists a subsequence (q_{m_j}) of (q_m) and $q \in E$ such that $q_{m_j} \rightharpoonup q$ in E. We may also assume that (q_m) is a normalized sequence. By Sobolev imbedding theorem we have $q_{m_j} \to q$ in $L^{loc}_{\infty}(\mathbf{R}, \mathbf{R}^n)$. Hence $q \neq 0$. Now

$$\begin{aligned} 0 = \langle I'(q_{m_j}), \varphi\rangle = &\int_{-\infty}^{\infty} (\langle \dot{q}_{m_j}, \dot{\varphi}\rangle + \langle \varphi, L(t)q_{m_j}\rangle)dt \\ &- \int_{-\infty}^{\infty} \langle \varphi, V_q(t, q_{m_j})\rangle dt. \end{aligned}$$

By taking limits we obtain

$$\begin{aligned}0 &= \int_{-\infty}^{\infty} (\langle \dot{q}, \dot{\varphi} \rangle + \langle \varphi, L(t)q \rangle)dt - \int_{-\infty}^{\infty} \langle \varphi, V_q(t,q) \rangle dt \\ &= \langle I'(q), \varphi \rangle.\end{aligned}$$

Hence q is a critical point of I. Let $w_m = q_{m_j} - q$. Then as in Proposition 1.2 in [**3**] we can show that

$$\begin{aligned}I(w_m) &\to c - I(q), \\ I'(w_m) &\to 0.\end{aligned}$$

Now

$$I(w_m) = \frac{1}{2} \int_{-\infty}^{\infty} (|\dot{w}_m|^2 + \langle w_m, L(t) w_m \rangle)dt - \int_{-\infty}^{\infty} V(t, w_m)dt$$

and

$$\begin{aligned}\langle I'(w_m), w_m \rangle = &\int_{-\infty}^{\infty} (|\dot{w}_m|^2 + \langle w_m, L(t) w_m \rangle)dt \\ &- \int_{-\infty}^{\infty} \langle w_m, V_q(t, w_m) \rangle dt.\end{aligned}$$

Hence

$$\begin{aligned}I(w_m) - \frac{1}{2}\langle I'(w_m), w_m \rangle &\geq \left(\frac{\mu}{2} - 1\right) \int_{-\infty}^{\infty} V(t, w_m)dt \\ &\geq 0.\end{aligned}$$

Thus

$$0 \leq I(w_m) - \frac{1}{2}\langle I'(w_m), w_m \rangle \leq I(w_m) + M\|I'(w_m)\|$$

for some constant M independent of m. Therefore

$$0 \leq c - I(q).$$

Since $c = \bar{c} = \inf_{q \in \mathcal{K}\setminus\{0\}} I(q)$, this completes the proof. □

The following fact is crucial to the existence of infinitely many homoclinic solutions of (HS).

LEMMA 1.5. *Let $q \in E$ be a critical point of I with $I(q) = c$. Choose $\overline{q}$ on the ray passing through 0 and q such that $I(\overline{q}) < 0$. Define a function $g : [0,1] \to E$ by $g(\theta) = \theta\overline{q}$. Then*

(1) $g \in \Gamma$,
(2) $Max_{\theta\in[0,1]} I(g(\theta)) = c$, *and*
(3) *for each $r > 0$, there exists $\varepsilon > 0$ such that $I(g(\theta)) > c - \varepsilon$ implies $g(\theta) \in B_r(q)$.*

Proof. (i) and (ii) are evident from the construction of g and (V_4). Suppose $q = \overline{\theta}\overline{q}$, $0 < \overline{\theta} < 1$. Then for any $\varepsilon > 0$, by (V_4), there are constants $\theta_{-\varepsilon}$ and $\theta_{+\varepsilon}$ with $\theta_{-\varepsilon} < \overline{\theta} < \theta_{+\varepsilon}$ such that $\theta_{\pm\varepsilon} \to \overline{\theta}$ as $\varepsilon \to 0$ and $I(\theta\overline{q}) > c - \varepsilon$ if and only if $\theta \in (\theta_-, \theta_+)$. In particular for each $r > 0$ there is an $\varepsilon = \varepsilon(r)$ such that $\theta \in (\theta_-, \theta_+)$ implies that $g(\theta) = \theta\overline{q} \in B_r(q)$. □

At this point assume further that V satisfies one further condition

(**) There is an $\alpha > 0$ such that $\mathcal{K}^{c+\alpha}$consists of isolated points.

Observe that the above proposition corresponds to Proposition 2.22 [**3**].Therefore we can apply the argument in [**3**] to prove the existence of infinitely many homoclinic solutions of (HS). Therefore the following theorem was essentially proved in [**3**].

THEOREM 1.3. *If V satisfies (V_1)-(V_4) and (**) ,then the problem (HS) has infinitely many homoclinic solutions*

2. Main result

The condition (*) ,which asserts the finiteness of normalized critical points of V in $I^{c+\alpha}/\mathbf{Z}$, was required to use the special property of the space $E = W^{1,2}(\mathbf{R}, \mathbf{R}^n)$. So it could be replaced with the condition (**), which asserts the discreteness of normalized critical points of V in $\mathcal{K}^{c+\alpha}$,if V satisfies the condition (V_4). Therefore it is natural to seek a condition on V which guarantees the the discreteness of critical points of V. We do this in the following Theorem.

THEOREM 2.1. *If V satisfies the conditions (V_1)-(V_4), and*

$$(\mathrm{V}_5) \qquad \langle V_{qq}(t,q)p,p\rangle \geq \kappa |p|^2, \quad p,q \in \mathbf{R}^n, \quad \kappa > -\frac{1}{2}$$

then the critical points of I are all isolated. Therefore the problem (HS) has infinitely many homoclinic solutions.

Proof. Let q be a critical point of I. Thus for any $p \in E$ we have

$$\begin{aligned} 0 = \langle I'(q),p\rangle = & \int_{-\infty}^{\infty} (\langle \dot q, \dot p\rangle + \langle p, L(t)q\rangle)dt \\ & - \int_{-\infty}^{\infty} \langle p, V_q(t,q)\rangle dt. \end{aligned}$$

Now

$$\begin{aligned} I(q+p) =& \frac{1}{2}\int_{-\infty}^{\infty} (|\dot q + \dot p|^2 + \langle q+p, L(t)(q+p)\rangle)dt - \int_{-\infty}^{\infty} V(t,q+p)dt \\ =& \frac{1}{2}\int_{-\infty}^{\infty} (|\dot q|^2 + \langle q, L(t)q\rangle)dt + \int_{-\infty}^{\infty} (\langle \dot q,\dot p\rangle + \langle p, L(t)q\rangle)dt \\ & + \frac{1}{2}\int_{-\infty}^{\infty} (|\dot p|^2 + \langle p, L(t)p\rangle)dt - \int_{-\infty}^{\infty} V(t,q+p)dt \\ =& I(q) + \int_{-\infty}^{\infty} V(t,q)dt + \int_{-\infty}^{\infty} \langle p, V_q(t,q)\rangle dt \\ & + \frac{1}{2}\|p\|^2 - \int_{-\infty}^{\infty} V(t,q+p)dt \\ =& I(q) + \frac{1}{2}\|p\|^2 + \int_{-\infty}^{\infty} (V(t,q) + \langle p, V_q(t,q)\rangle - V(t,q+p))dt. \end{aligned}$$

Now

$$\begin{aligned} & V(t,q) + \langle p, V_q(t,q)\rangle - V(t,q+p) \\ = & \int_0^1 s\langle p, V_{qq}(t,q+sp)p\rangle dt \\ = & \int_0^1 s\langle p, (V_{qq}(t,q+sp) - V_{qq}(t,q))p\rangle dt + \langle p, V_{qq}(t,q)p\rangle. \end{aligned}$$

Note that $\|p\|_{L^\infty} \leq \sqrt{2}\|p\|$. Hence

$$\int_{-\infty}^{\infty} (V(t,q) + \langle p, V_q(t,q)\rangle - V(t,q+p))dt$$
$$= o(\|p\|^2) + \int_{-\infty}^{\infty} \langle p, V_{qq}(t,q)p\rangle dt.$$

Observe that

$$\int_{-\infty}^{\infty} \langle p, V_{qq}(t,q)p\rangle dt = \|p\|^2 \int_{-\infty}^{\infty} \langle \frac{p}{\|p\|}, V_{qq}(t,q)\frac{p}{\|p\|}\rangle dt$$
$$= h(p)\|p\|^2.$$

We see here that h is homogeneous of degree 0 and that $h \geq \kappa > -\frac{1}{2}$ by (V_5). Hence we now have the following estimate;

$$I(q+p) = I(q) + (\frac{1}{2} + h(p))\|p\|^2 + o(\|p\|^2).$$

This completes the proof. □

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