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# PROPERTIES OF THE CONTINUATION OF COMPLEX ANALYTIC MANIFOLDS SPREAD IN $C^n$

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## 0. Introduction

In the theory of one complex variable, it is well known that holomorphic functions cannot in general be extended as holomorphic functions to larger domains. In the space $C^n$ of $n$ complex variables, the conditions are different. In $C^n$, a domain of holomorphy has important meanings and is a fruitfull domain in which many function theoretical results are obtained.

For functions of two variables, we know the following result : If $\Omega = \{(z_1, z_2) : 1 < |z_1|^2 + |z_2|^2 < 4\}$ then all holomorphic functions on $\Omega$ can be extended to the ball $\tilde{\Omega} = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 < 4\}$. This means that zeros of holomorphic functions of more than one variable cannot be isolated.

H. J. Bremmermann [1], F. Norguet [5] and K. Oka [6] solved this problem in $C^n$ and in the unramified domain over $C^n$, F. Docquier and H. Grauert [3] solved this problem in Stein manifold , and J. Kajiwara and K. H. Shon [4] obtained the continuation and vanishing theorem for cohomology of infinite dimensional spaces. We will introduce some properties of the continuation of holomorphic functions and a domain of holomorphy.

## 1. Prelimirary and notation

Let $D$ be a Hausdorff topological space, $\varphi$ be a local homeomorphism on $D$ into $C^n = \{z = (z_1, z_2, \cdots, z_n) | z_j \in C, 1 \leqq j \leqq n\}$ and $(D, \varphi)$ be a (Riemann) domain over $C^n$ i.e. $\varphi$ spread of $D$ in $C^n$.

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DEFINITION 1.1. A domain over $C^n$ is a pair $(D, \varphi)$ with the following properties:

1) $D$ is a connected topological space.

2) For every two points $x_1, x_2 \in D$ with $x_1 \neq x_2$ there are open neighborhoods $U_1 = U_1(x_1) \subset D, U_2 = U_2(x_2) \subset D$ with $U_1 \cap U_2 = \emptyset$.

3) $\varphi : D \longrightarrow C^n$ is a locally homeomorphism.

DEFINITION 1.2. A triple $(\lambda, D', \varphi')$ is an analytic completion of a domain $(D, \varphi)$ over $C^n$ if

1) $(D', \varphi')$ is a domain over $C^n$ i.e. $\varphi' : D' \longrightarrow C^n$ is a locally homeomorphism.

2) $\lambda : D \longrightarrow D'$ is a local homeomorphism.

3) $\varphi = \varphi' \circ \lambda$.

4) For any holomorphic function $f$ on $D$, there is a holomorphic function $f'$ on $D'$ such that $f = f' \circ \lambda$.

We say that the above described diagram is commutative if $\varphi = \varphi' \circ \lambda$ and $f = f' \circ \lambda$, and we say that $f'$ is an analytic prolongation of $f$ to $(\lambda, D', \varphi')$.

In case that $n = 1$, there is no analytic completion. In case of $n \geqq 2$, we let $z = (z_1, z_2, \cdots, z_n), z' = (z_2, z_3, \cdots, z_n)$ and $z = (z, z') \in D$ where

$$D = \{z \in C^n | r^2 < |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 < R^2\}.$$

If $z$ is a point of $D$ then

$$\sqrt{r^2 - |z_2|^2 - |z_3|^2 - \cdots - |z_n|^2} < |z_1| < \sqrt{R^2 - |z_2|^2 - |z_3|^2 - \cdots - |z_n|^2}.$$

For $z' \in C^{n-1}$ with $|z_2|^2 + |z_3|^2 + \cdots + |z_n|^2 > r^2$ we have

$$|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 > r^2$$

and

$$z \in D \Leftrightarrow |z_1| < \sqrt{R^2 - |z_2|^2 - |z_3|^2 - \cdots - |z_n|^2}.$$

In this case,

$$\begin{aligned}\Delta(z') =&\{z_1 \in C | \sqrt{r^2 - |z_2|^2 - |z_3|^2 - \cdots - |z_n|^2} < |z_1| \\ &< \sqrt{R^2 - |z_2|^2 - |z_3|^2 - \cdots - |z_n|^2}\} \\ =&\{z_1 \in C | |z_1| < \sqrt{R^2 - |z_2|^2 - |z_3|^2 - \cdots - |z_n|^2}\}.\end{aligned}$$

For $z' \in C^{n-1}$ with $|z_2|^2 + |z_3|^2 + \cdots + |z_n|^2 \leqq r^2$, $\Delta(z')$ is the of form

$$\{z \in C | 0 < |z_1| < \sqrt{R^2 - |z_2|^2 - |z_3|^2 - \cdots - |z_n|^2}\}$$

or

$$\begin{aligned}\{z \in C | &\sqrt{r^2 - |z_2|^2 - |z_3|^2 - \cdots - |z_n|^2} < |z_1| \\ &< \sqrt{R^2 - |z_2|^2 - |z_3|^2 - \cdots - |z_n|^2}\}.\end{aligned}$$

Thus we obtain a well defined holomorphic function $f'$ on $D' = \{z \in C^n | |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 < R^2\}$. Therefore, a holomorphic function of $n \geqq 2$ complex variables cannot have isolated singularities. Furthermore, it cannnot have isolated zeros.

DEFINITION 1.3. Let $(D, \varphi)$ be a domain over a (connected) complex analytic manifold $M$ and $\mathcal{H}(D)$ be a family of holomorphic functions on $D$. A triple $(\tilde{\lambda}, \tilde{D}, \tilde{\varphi}) = (\tilde{D}, \tilde{\varphi}, \tilde{\lambda}, \tilde{f})$ is an envelope of holomorphy ( or a continuation) of $(D, \varphi) = (D, \varphi, f)$ with respect to $\mathcal{H}(D)$ if

1) $(\tilde{\lambda}, \tilde{D}, \tilde{\varphi})$ is an analytic completion of $(D, \varphi)$ with respect to $\mathcal{H}(D)$.

2) For any analytic completion $(\tilde{\lambda}, \tilde{D}, \tilde{\varphi})$ of $(D, \varphi)$, there is a local homeomorphism $\mu : D' \longrightarrow \tilde{D}$ such that $(\mu, \tilde{D}, \tilde{\varphi})$ is an analytic completion of $(D', \varphi')$ with respect to the family of all analytic continuations of functions of $\mathcal{H}(D)$.

Let $V$ be a complex (analytic) manifold and $\mathcal{H}(V)$ be a family of holomorphic functions on $V$. A triple $(\lambda, V', \varphi')$ is an envelope of holomorphy of $(V, \varphi)$ with respect to $\mathcal{H}(V)$ if

1) $(V', \varphi')$ is a domain over $C^n$.

2) For any $f$ in $\mathcal{H}(V)$, there is a holomorphic function $f'$ on $V'$ such that $f = f' \circ \lambda$ and $\varphi = \varphi' \circ \lambda$.

3) $\lambda : V \longrightarrow V'$ is a local homeomorphism.

## 2. Geometrically properties of complex domains

H. Cartan [2] proved the unique existence of such envelope of holomorphy. Espectially if $\mathcal{H}(D)$ consists of only one holomorphic function $f$ on $D$, the envelope of holomorphy of $(D,\varphi)$ with respect to $\mathcal{H}(D)$ is called a domain of holomorphy of $f$. A domain over $M$ which is a domain of holomorphy of a holomorphic function on a domain over $M$ is called shortly a domain of holomorphy. Moreover if $\mathcal{H}(D)$ is the family of all holomorphic functions on $D$, the envelope of holomorphy of $(D,\varphi)$ with respect to $\mathcal{H}(D)$ is called shortly the envelope of holomorphy of $(D,\varphi)$.

DEFINITION 2.1. The maximal continuation of $(D,\varphi,\mathcal{H}(D))$ is called the envelope of holomorphy of $(D,\varphi)$.

THEOREM 2.2. *Let $V$ be a complex manifold, $(V,\varphi_j)(j=1,2)$ be two domains over $C^n$, and let $(V_j,\mu,F_j,\mathcal{H}(V_j))$ be the maximal continuation of $(V,\varphi_j,\mathcal{H}(V))$ for $j=1,2$. Then $V_1,V_2$ are homeomorphic under a mapping $G:V_2\longrightarrow V_1$ such that the following diagram*

$$\begin{array}{ccccc} C^n & \overset{\varphi_1}{\longleftarrow} & V & \overset{\varphi_2}{\longrightarrow} & C^n \\ \mu_1\uparrow & & & & \uparrow\mu_2 \\ V_1 & & \overset{G}{\longleftarrow} & & V_2 \end{array}$$

*with local homeomorphisms $F_1:V\longrightarrow V_1$ and $F_2:V\longrightarrow V_2$ is full commutative.*

*proof.* If $\varphi_1=(\varphi_{11},\varphi_{12},\cdots,\varphi_{1n})$, then the functions $\varphi_{1j}\in\mathcal{H}(V)$ and so can be continued to functions $\varphi_1^*=(\varphi_{11}^*,\varphi_{12}^*,\cdots,\varphi_{1n}^*)$ on $V_2$. Since $\varphi_1,\varphi_2$ are local homeomorphic, the Jacobian $J=\dfrac{\partial\varphi_2}{\partial\varphi_1}$ never vanishes on $V$ and $J\in\mathcal{H}(V)$. Both $J$ and $J^{-1}$ can be continued to functions $J^*$ and $(J^{-1})^*$ in $\mathcal{H}(V_2)$. By the principle of analytic continuation we have $J\circ J^{-1}\equiv 1$ on $V$. Hence $J^*(J^{-1})^*\equiv 1$ on $V_2$. Therefore $J^*$ never vanishes. And $J^*=\dfrac{\partial\varphi_1^*}{\partial\mu_2}$, so that $\varphi_1^*$ is a local homeomorphism and $(V_2,\varphi_1^*)$ is a domain over $C^n$. Since $\varphi_1=\varphi_1^*\circ F_2$, $(V_2,\varphi_1^*,F_2,\mathcal{H}(V_2))$ is a continuation of $(V,\varphi,\mathcal{H}(M))$, and maximality of $(V_1, \mu_1, F_1, \mathcal{H}(V_1))$ implies the existence of a domain $(V_2, G)$ over $V_1$ such that the following diagram commutes.

$$\begin{array}{ccccc} V & \xrightarrow{F_1} & V_1 & \xrightarrow{\mu_1} & C^n \\ \searrow_{F_2} & & \uparrow G & & \nearrow_{\varphi_1^*} \\ & & V_2 & & \end{array}$$

Similarly extending $\varphi_2$ to $\varphi_2^*$ we find $\varphi_2^*$ is a local homeomorphic spreading $V_1$ in $C^n$ such that $\varphi_2 = \varphi_2^* \circ F_1$. If we use maximality of $(V_2, \mu_2, F_2, \mathcal{H}(V_2))$, we obtain a mapping $\tilde{G}$ which spread $V_1$ in $V_2$ and gives the following commutative diagram.

$$\begin{array}{ccccc} V & \xrightarrow{F_2} & V_2 & \xrightarrow{\mu_2} & C^n \\ \searrow_{F_1} & & \uparrow \tilde{G} & & \nearrow_{\varphi_2^*} \\ & & V_1 & & \end{array}$$

Hence the above diagram with $\varphi_2 : V \longrightarrow C^n$ is commutative. If $a \in V$ and $U$ is a small neighborhood of $a$, then $F_2$ is homeomorphic on $U$ into $V_2$. On $F_2(U)$, we have

$$\tilde{G}G(y) = y \Leftrightarrow \tilde{G}G(F_2(x)) = F_2(x)$$

for all $x \in U$. But from the above diagrams, we know that

$$F_2^{-1}\tilde{G}GF_2(x) = F_2^{-1}\tilde{G}F_1(x) = F_2^{-1}F_2(x) = x.$$

Therefore, from the analytic continuation, $\tilde{G}G(y) = y$ on $V_2$. Similarly, $G\tilde{G}(x) = x$ on $V_1$ and $G$ is global homeomorphic.

COROLLARY 2.3. *The envelope of holomorphy is not univalent.*

PROPOSITION 2.4. *Let* $M = \{(z_1, z_2) \in C^2 \| z_2| < Re(z_1) < |z_2| + 6\}$ *and*

$$\varphi : (z_1, z_2) \to (e^{iz_1}, z_2).$$

*If $\tilde{M} = \varphi(M)$, then $\varphi$ is a homeomorphism of $M$ onto $\tilde{M}$. Any $f \in \mathcal{H}(M)$ can be extended to $N = \{(z_1, z_2) \in C^2 | |z_2| < Re(z_1)\}$.*

*Proof.* We know the first part from $6 < 2\pi$. If $\mathrm{Re}(z_1) > 0$, then $f(z, \zeta)$ is defined and holomorphic for all $\zeta$ with $\mathrm{Re}(z_1) - 6 < |\zeta| < \mathrm{Re}(z_1)$, and we see that $f$ extends by examinig the coefficient functions $\varphi_k(z_1)(k < 0)$ in the Laurent expansion $f(z_1, \zeta) = \sum_{k=-\infty}^{\infty} \varphi_k(z_1)\zeta^k$. But $N$ is a domain of holomorphy because it is convex, and this means that $(N, \varphi, \tilde{\mathcal{H}}, i_1)$ is the envelope of holomorphy of $(M, \varphi, \mathcal{H}(M))$ where $i_1 = \mathrm{id}_M$ and

$$\tilde{\mathcal{H}} = \{\tilde{f} \in \mathcal{H}(N) | \ \tilde{f} \text{extends } f \text{ to } N, f \in \mathcal{H}(M)\}$$

and $\varphi$ is identified with the natural spread of the envelope of holomorphy in $C^n$.

REMARK 2.5. If $U \subset M$ is a compact set, and $\tilde{U} = \imath_1(U)$ its images in $N$ then $\varphi^{-1}(\varphi(\tilde{U}))$ is usually not compact.

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