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BRIESKORN MANIFOLD INDUCED IN HYPERSURFACE OF A PRODUCT OF TWO SPHERES

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0. Introduction

The *(2n-1)-dimensional Brieskorn manifold* B^{2n-1} is the intersection of a complex cone and a $(2n+1)$-dimensional unit sphere $S^{2n+1}(1)$.

It is well-known that every Brieskorn manifold is a contact manifold and admits many Sasakian structures[1].

The purpose of the present paper is to show that the converse statement is true in hypersurface of a product of two spheres under some additional condition.

In section 1, we recall the structure equations of hypersurface M of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.

In section 2, we prove that the converse statement is true in M. All geometric objects and mappings throughout this paper are of C^∞.

1. Structure equations of hypersurface of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$

K.Yano[2] proved that the (f, g, u, v, λ)-structure is natually induced on $S^n \times S^n$ as a submanifold of codimension 2 of a $(2n+2)$-dimensional Euclidean space or a real hypersurface of $(2n+1)$-dimensional unit sphere $S^{2n+1}(1)$.

S.-S. Eum,U-H. Ki and Y.H. Kim [3] researched the condition that a real hypersurface of $S^n \times S^n$ becomes a Brieskorn manifold.

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Let M be a hypersurface immersed isometrically in $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ and suppose that M is covered by the system of coordinate neighborhoods $\{\bar{V}; \bar{x}^a\}$, where here and in the sequel, the indices $a, b, c, d, \cdots$ run over the range $\{1, 2, \cdots, 2n-1\}$.

Since the immersion $i : M \to S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ is isometric from the (f, g, u, v, λ)-structure defined on $S^n \times S^n$, we obtain the so called $(f, g, u, v, w, \lambda, \mu, \nu)$-structure[2] given by

$$\begin{aligned} f_b^e f_e^a &= -\delta_b^a + u_b u^a + v_b v^a + w_b w^a, \\ f_e^a u^e &= -\lambda v^a + \mu w^a, \\ f_e^a v^e &= \lambda u^a + \nu w^a, \\ f_e^a w^e &= -\mu u^a - \nu v^a \end{aligned} \tag{1.1}$$

or, equivalently

$$\begin{aligned} &u_e f_a^e = \lambda v_a - \mu w_a, v_e f_a^e = -\lambda u_a - \nu w_a, w_e f_a^e = \mu u_a + \nu v_a, \\ &u_e u^e = 1 - \lambda^2 - \mu^2, u_e v^e = -\mu\nu, u_e w^e = -\lambda\nu, \\ &v_e v^e = 1 - \mu^2 - \nu^2, v_e w^e = \lambda\mu \\ &w_e w^e = 1 - \mu^2 - \nu^2 \end{aligned} \tag{1.2}$$

where u_a, v_a and w_a are 1-forms associated with u^a, v^a and w^a respectively given by $u_a = u^b g_{ba}, v_a = v^b g_{ba}$ and $w_a = w^b g_{ba}$, and $f_{ba} = f_b^c g_{ca}$ is skew-symmetric. Moreover, we get

$$k_c^e k_e^a = \delta_c^a - k_c k^a, \tag{1.3}$$

$$k_e k^e = 1 - \alpha^2, \tag{1.4}$$

$$k_c^e u_e = -v_c - \mu k_c, k_c^e v_e = -u_c - \nu k_c, \tag{1.5}$$

$$\nabla_c u_b = \mu l_{cb} - \lambda k_{cb} + f_{cb}, \tag{1.6}$$

$$\nabla_c v_b = k_c^e f_{eb} - k_c w_b + \nu l_{cb} + \lambda g_{cb}. \tag{1.7}$$

We introduce the following Remark and Theorem for later use.

REMARK [4]. *If $\lambda^2+\mu^2+\nu^2=1$ on the hypersurface M, we see that*

$$\mu=0,\quad \nu=constant(\neq 0),\quad v_c=0\quad \text{and}\quad \alpha=0.$$

And if the function λ vanishes on some open set, then we have $v_c=0$ and $\mu=0$. Moreover the 1-form u_b never vanishes on an open set in M, in fact, if the 1-form u_b is zero on an open set in M, then $f_{cb}=0$, which contradicts $n>1$.

THEOREM A [4]. *Let M be a hypersurface of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})(n>1)$ with $(f,g,u,v,w,\lambda,\mu,\nu)$-structure satisfying $\lambda^2+\mu^2+\nu^2=1$. If M is minimal, then M as a submanifold of codimension 3 of a $(2n+2)$-dimensional Euclidean space E^{2n+2} is an intersection of a complex cone with generator C and $(2n+1)$-dimensional unit sphere $S^{2n+1}(1)$.*

2. Brieskorn manifold induced in Hypersurface

In this section, we assume that the $(f,g,u,v,w,\lambda,\mu,\nu)$-structure induced on M admits *an almost contact metric structure* (f_b^a, g_{cb}, p^a) and $n>1$, that is , M satisfies

$$f_b^e f_e^a = -\delta_b^a + p_b p^a, \tag{2.1}$$

$$p_a p^a = 1. \tag{2.2}$$

From (1.1) and (2.1), we see

$$p_b p^a = u_b u^a + v_b v^a + w_b w^a. \tag{2.3}$$

Contracting this equation with respect to the indices a and b, we get

$$p_a p^a = 1 + 2(1-\lambda^2-\mu^2-\nu^2)$$

with the aid of (1.2).

Therefore, we have

LEMMA 2.1. *Let M be a hypersurface of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})(n>1)$ with $(f,g,u,v,w,\lambda,\mu,\nu)$-structure. Then M admits an almost contact metric structure (f_b^a, g_{cb}, p^a) if and only if $\lambda^2+\mu^2+\nu^2=1$ holds on M.*

Using Remark stated in section 1, (1.4) becomes

$$k_e k^e = 1, \tag{2.4}$$

so k^a is an unit vector, and it follows from (1.5) that

$$u_c = -\nu k_c, \tag{2.5}$$

and the second equation of (1.2) and (2.3) yields

$$w_c = \lambda k_c. \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.3), and taking account of Remark and Lemma 2.1, we find

$$p_b p^a = k_b k^a,$$

or equivalently

$$p_b = \pm k_b. \tag{2.7}$$

Thus we have

LEMMA 2.2. *Let M be a hypersurface of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})(n>1)$ with $(f,g,u,v,w,\lambda,\mu,\nu)$-structure. If M admits an almost contact metric structure (f_b^a, g_{cb}, p^a) , then the aggregate (f_b^a, g_{cb}, k^a) also defines an almost contact metric structure.*

Transvecting g^{cb} to (1.7) and taking account of Remark , (2.4) and (2.6), we find

$$l = -2\lambda(n-1)/\nu, \tag{2.8}$$

where we have put $l = g^{cb} l_{cb}$.

Hence, we have

LEMMA 2.3. *Let M be a hypersurface of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})(n > 1)$ with $(f, g, u, v, w, \lambda, \mu, \nu)$-structure. If M admits an almost contact metric structure (f_b^a, g_{cb}, p^a) , then M has a constant mean curvature.*

Next we assume that p^c is a killing vector on M. Since we have $\mathcal{L}_p g_{cb} = 0$ for a killing vector p^a, we have also $\mathcal{L}_k g_{cb} = 0$, and

$$\mathcal{L}_u g_{cb} = \nabla_c u_b + \nabla_b u_c = -2\lambda k_{cb} = 0, \tag{2.9}$$

with the aid of (1.6) and Remark.

Transvecting g^{cb} to (2.9) and taking account of (1.3), (2.4) and $n > 1$, we obtain

$$\lambda = 0. \tag{2.10}$$

From (2.8) and (2.10), it follows that M is minimal.

Conversely, if M is minimal, then the equation (1.6) together (2.8), (2.10) and Remark implies that

$$\nabla_c u_b = f_{cb},$$

which shows that u^c is a killing vector because f_{cb} is skew-symmetric with respect to b and c, and also both k^a and p^a are killing vectors because of (2.7) and (2.5).

Thus we have

THEOREM 2.4. *Let M be a hypersurface of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})(n > 1)$ with $(f, g, u, v, w, \lambda, \mu, \nu)$-structure, and let M admits an almost contact metric structure (f_b^a, g_{cb}, p^a). Then p^a is a killing vector if and only if M is minimal.*

And also, by making use of Theorem 3.1 [5] and (2.10), we have

THEOREM 2.5. *Under the same assumptions as those stated in Theorem 2.4. If p^a is a killing vector, then M is an f-invariant hypersurface.*

Combining Lemma 2.1 and Theorem 2.4 with Theorem A , we conclude

THEOREM 2.6. *Let M be a hypersurface of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})(n > 1)$ with $(f,g,u,v,w,\lambda,\mu,\nu)$-structure, and let M admits an almost contact metric structure (f_b^a, g_{cb}, p^a), p^a being a killing vector. Then M as a submanifold of codimension 3 of a $(2n+2)$-dimensional Euclidean space E^{2n+2} is an intersection of a complex cone with generator C and a $(2n+1)$-dimensional unit sphere $S^{2n+1}(1)$, that is, a Brieskorn manifold B^{2n-1}.*

Finally, let M admit a *Sasakian structure*, that is, the given structure satisfies (2.1), (2.2) and

$$\nabla_c f_b^a = -g_{cb}p^a + \delta_c^a p_b. \tag{2.11}$$

Transvecting (2.1) with $p_a p^b$ and making use of (2.2), we get

$$f_b^e p_e = 0. \tag{2.12}$$

If we differentiaite (2.2) covariantly, we obtain

$$(\nabla_c p^e)p_e = 0. \tag{2.13}$$

Differentiating (2.12) covariantly and using (2.11), we find

$$f_b^e(\nabla_c p_e) = g_{cb} - p_c p_b.$$

Applying this with f_a^b, and making use of (2.1), (2.12) and (2.13), we have

$$\nabla_c p_a = f_{ca},$$

which implies that p^a is a killing vector.

Combining Theorem 2.6 with the fact that p^a is a killing vector, we find

THEOREM 2.7. *Let M be a hypersurface of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})(n > 1)$ with $(f,g,u,v,w,\lambda,\mu,\nu)$-structure. If M admits a Sasakian structure, then M is a Brieskorn manifold B^{2n-1}.*

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