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THE EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR LINEAR RETARDED FUNCTIONAL DIFFERENTIAL EQUATION IN HILBERT SPACE

WEON-KEE KANG AND DONG-GUN PARK

1. Introduction

This paper is concerned with the existence and uniqueness of solutions on the delay form functional differential equation

$$\frac{du(t)}{dt} + Au(t) + A_1u(t-h) + \int_{-h}^{0} a(-s)A_2u(t+s)ds = 0, \quad 0 \le t \le T \tag{1.1}$$

$$u(0) = x,\ u(s) = y(s), \qquad s \in [-h, 0) \tag{1.2}$$

in a complex Hilbert space H, where $a(-s)$ is a complex valued function of bounded variation over an interval $[-h, 0]$.

Let H and V be complex Hilbert spaces such that V is a dense subspace of H and the inclusion mapping V into H is continuous. The norms of H and V are denoted by $|\cdot|$ and $\|\cdot\|$, respectively. Identifying H with its antidual we may write $V \subset H \subset V^*$. For a couple of Hilbert space V and H the notation $B(V, H)$ denotes the totality of bounded linear mappings of V into H, and $B(H) = B(H, H)$.

Here, A is the operator associated with a sesquilinear form $a(u, v)$ which is defined in $V \times V$ and satisfies Gårding's inequality

$$\text{Re } a(u, u) \ge c\|u\|^2.$$

Let A_1 and A_2 be operators in $B(V, V^*)$.

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Equations of the type (1.1) were invetigated by G. Di Blasio, K. Kunisch and E. Sinestrari [2], [3], E. Sinestrari [11] and H. Tanabe [12].

In [2] the initial value problem for the equations in Hilbert space H was solved in the space of L^2 functions with values in H. Essential use was made of the maximal regularity result for equations without delay terms there, and the corresponding regularity result was also obtained for the equations with delay terms. In [3] stability results were established for equations in Hilbert space. In [11] equations in a general Banach space E were investigated without assuming that A is densely defined. The solvability was established in the space maximal regularity results. In H.Tanabe [12] the initial value problem for the equations (1.1) in a Banach space X was constructed the fundamental solution in the sense of S. Nakagiri [8], [9]. It was shown that the mild solution satisfying the initial condition $u(s) = y(s)$, $s \in [-h, 0)$ expressed by S. Nakagiri's formula is actually the strict solution of (1.1), (1.2) provided that f is a Hölder continuous function in $[-h, 0]$ with values in the Banach space $D(A)$ endowed with the graph norm of A but with no maximal regularity result. An example of such a function $f \notin L^1(0, T : H)$ is given in the appendix.

M. G. Crandall and J. A. Nohel [5] study the existence, uniqueness, regularity and dependence upon data of a strong solutions of abstract functional differential equation

$$\frac{du}{dt} + Au \ni Gu \quad (0 \le t \le T)$$
$$u(0) = x$$

in a real Banach space.

In the proof of the main result, we can transformed (1.3) to (1.1).

$$u'(t) + Au(t) + \int_0^t a(t-s)A_2u(s)ds = f(t) \tag{1.3}$$

where $f(t) = -A_1y(t-h) - \int_{t-h}^0 a(t-s)A_2y(s)ds$.

With the aid of a method to [5] we can transformed (1.4) to (1.3).

$$u'(t) + Au(t) = G(u)(t), \tag{1.4}$$

$$G(u) = f + R*f - R(0)u + Rx - \dot{R}*u \tag{1.5}$$

where the notation $(a * b)(t) = \int_0^t a(t-s)b(s)ds$. The function R is of bounded variation with values in $B(H)$ as well as in $B(V^*)$, and $G(u)$ will be considered as a function with values in H and also in V^* for $u \in C([0,T] : H)$.

2. Assumptions and Main Theorem

Let $a(u,v)$ be a sesquilinear form defined on $V \times V$. Suppose that there exist positive constants C and c such that

$$(2.1) \qquad |a(u,v)| \le C\|u\|\|v\|, \quad \mathrm{Re}\, a(u,u) \ge c\|u\|^2$$

for any $u, v \in V$. Let $A \in B(V, V^*)$ be the operator associated with this sesquilinear form: $(Au, v) = a(u,v)$, for any $u, v \in V$. The realization of A in H which is the restriction of A to $D(A) = \{u \in V : Au \in H\}$ is also denoted by the same letter A. For the sake of convenience we assume that A has an everywhere defined bounded inverse. The sesquilinear form $\overline{a(u,v)}$ is called the adjoint sesquilinear form of $a(u,v)$. Let A^* be the adjoint of A. We assume that there exist a positive constant C such that

$$|a(u,v) - \overline{a(u,v)}| \le C\|u\|\|v\|.$$

Thus, we have

$$(2.2) \qquad |(A^* - A)u| \le C\|u\|.$$

Let A_i, $(i = 1,2)$ be operators in $B(V, V^*)$. Then $A_i A^{-1} \in B(V^*)$, for $i = 1, 2$. We assume also that $A_i A^{-1} \in B(H)$, $(i = 1,2)$.

We assume

$$(2.3) \qquad x \in H$$

$$(2.4) \qquad y \in L^2(-h, 0 : V) \cap L^2(-h, 0 : D(A), (s+h)ds)$$

where $y \in L^2(-h, 0 : V)$ and $y \in L^2(-h, 0 : D(A), (s+h)ds)$ mean $\int_{-h}^0 |y(s)|^2 ds < +\infty$ and $\int_{-h}^0 |Ay(s)|^2 (s+h) ds < +\infty$, respectively.

For $-h < \sigma < \tau \le 0$, it follows that

$$\int_\sigma^\tau Ay(s)ds = \int_\sigma^0 Ay(s)ds - \int_\tau^0 Ay(s)ds.$$

Hence, we put

$$c_1 = \sup_{-h<\sigma<\tau\leq 0} |\int_\sigma^\tau Ay(s)ds| < +\infty.$$

We consider the existence and uniqueness of solutions of the abstract functional differential equation :

$$\frac{d}{dt}u(t) + Au(t) = G(u)(t) \quad , \quad 0 < t \leq T \tag{2.5}$$

$$u(0) = x. \tag{2.6}$$

According to M. G. Crandall and J. A. Nohel [5] it suffices to prove the following proposition in order to establish below.

DEFINITION 2.1. A strong solution u of (2.5) on $[0,T]$ is a function $u \in L^2(0,T:V) \cap L^2(0,T:D(A),tdt)$ such that (2.5) (2.6).

Our main theorem is the following.

THEOREM 2.2. *Let x and y satisfy (2.3) and (2.4). The solution u of (1.1) and (1.2) exists and is unique.*

PROPOSITION 2.3. *The equation (1.1) is equivalent to the linear Volterra integro-differential equation (2.5) over an interval [0,T]*

3. The Proof of Theorem 2.2

3.1. Construction of Solution in [0,h)

In the following we make formal calculation.
It is easy that the following :
if $t \in [0,h)$, then it follows that $-h \leq t-h < 0$, hence, the initial condition is $u(t-h) = y(t-h)$. We obtain

$$\int_{-h}^0 a(-s)A_2u(t+s)ds = \int_{t-h}^0 a(t-s)A_2u(s)ds + \int_0^t a(t-s)A_2u(s)ds.$$

We put

$$f(t) = -A_1y(t-h) - \int_{t-h}^0 a(t-s)A_2y(s)ds$$

by a variable transformation and an elementary calculation. Therefore, the equation (1.1) is equivalent to the Volterra equation (1.3).

PROPOSITION 3.1. *Let x and y satisfy (2.3) and (2.4) over an interval $[0,h)$. Then the function $f(t) \in L^2(0,h:V^*) \cap L^2(0,h:H,tdt)$ exists in H.*

Proof. Since

$$\int_{t-h}^{0} a(t-s)A_2 y(s)ds = A_2 A^{-1} \int_{t-h}^{0} a(t-s)Ay(s)ds$$
$$= A_2 A^{-1}\{a(t)\int_{t-h}^{0} Ay(\sigma)d\sigma - \int_{t-h}^{0}\int_{t-h}^{s} Ay(\sigma)d\sigma da(t-s)\}.$$

Hence, we obtain

$$|\int_{t-h}^{0} a(t-s)A_2 y(s)ds| \leq |A_2 A^{-1}| C_1 \{|a(t)| + V(a:-h,t)\}$$

where $V(a:-h,t)$ is the total variation of a on the interval $(-h,t]$. In view of the elementary calculation, we obtain

$$\left\{\int_0^h |f(t)|^2 tdt\right\}^{\frac{1}{2}}$$
$$\leq |A_1 A^{-1}| \left\{ [\int_{-h}^{0} |Ay(s)|^2 (s+h)ds]^{\frac{1}{2}} \right.$$
$$\left. + \sup_{0 \leq t < h} |\int_{t-h}^{0} a(t-s)A_2 y(s)ds| \frac{h}{\sqrt{2}} \right\} < +\infty.$$

We follow that

$$f(t) \in L^2(0,h:H,tdt).$$

And, we obtain that

$$\left(\int_0^h \|f(t)\|_*^2 dt\right)^{\frac{1}{2}} \leq \left(\int_0^h \|A_1 y(t-h)\|_*^2 dt\right)^{\frac{1}{2}}$$
$$+ \left\{\int_0^h \|\int_{t-h}^{0} a(t-s)A_2 y(s)ds\|_*^2 dt\right\}^{\frac{1}{2}} < +\infty$$

where $\|\cdot\|_*$ stands for the norm of V^*. Hence the proof is complete.

PROPOSITION 3.2. *If* $u \in C((0,h] : H)$, *then*

(1) $G(u) \in L^2(0,h : V^*) \cap L^2(0,T : H, tdt)$

(2) $\int_{+0}^{h} G(u)(\tau)d\tau = \lim_{\epsilon \to +0} \int_{\epsilon}^{t} G(u)(\tau)d\tau$ *exists in* H.

Proof. Let $R(t)$ is of bounded variation over an interval $(0,h]$. We have

$$|(R * f)(t)| = |\int_{+0}^{t} R(t-s)\frac{d}{ds}\int_{\epsilon}^{s} f(\sigma)d\sigma ds|$$
$$\leq |R(0)|| \int_{+0\_}^{t} f(s)|ds + V(R : 0,t) \max_{0 \leq s \leq t} | \int_{+0}^{s} f(\sigma)d\sigma|.$$

Hence, we have

$$R * f \in L^\infty(0,h : H) \subset L^2(0,h : V^*) \cap L^2(0,T : H, tdt).$$

If $u \in C((0,h] : H)$, then $R(0)u \in C((0,h) : H)$ is obvious. For any $x \in H$, we obtain $R(t)x \in L^\infty(0,h : H)$. Since

$$|(\dot{R} * u)(t)| = |\int_0^t d_s R(t-s)u(s)| \leq V(R : 0,t) \max_{0 \leq s \leq t} |u(s)|.$$

We get

$$\dot{R} * u \in L^\infty(0,h : H).$$

Hence, the proof is complete.

Let x and f be arbitrary element of H and $L^2(0,h : V^*)$, respectively. Then in view of Theorem of J.L. Lions [7] there exists a unique function $u \in L^2(0,T : V) \cap C([0,T] : H)$ satisfying

$$u^1 \in L^2(0,T : V^*) \tag{3.1}$$

$$\frac{d}{dt}u(t) + Au(t) = f(t) \tag{3.2}$$

$$u(0) = x \tag{3.3}$$

$$|u(t)|^2 + c\int_0^t \|u(s)\|^2 ds \leq |x|^2 + \frac{1}{c}\int_0^t \|f(s)\|_*^2 ds \tag{3.4}$$

where $\|\cdot\|_*$ stands for the norm of V^*.

PROPOSITION 3.3. *In addition to the above let $f \in L^2(0, h : H, tdt)$. Then the following inequality holds*

$$\int_0^t |u'(s)|^2 s ds \le (1 + \frac{C^2}{2c}t)|x|^2 + \frac{1}{c}(1 + \frac{C^2}{2c}t)\int_0^t \|f(s)\|_* ds + 2\int_0^t |f(s)|^2 s ds \tag{3.5}$$

Proof. In the following we make formal calculation. It is easy to justify it approximating x and f by nice elements.

$$\begin{aligned} &\frac{d}{dt}a(u(t), u(t)) = a(u'(t),\ u(t)) + a(u(t), u'(t)) \\ =&(u'(t), (A^* - A)u(t)) + (u'(t), Au(t)) + (Au(t), u'(t)) \\ =&2\mathrm{Re}(Au(t), u'(t)) + (u'(t), (A^* - A)u(t)) \end{aligned} \tag{3.6}$$

Taking inner product (3.2) and $u'(t)$, and using (3.6) we get

$$|u'(t)|^2 + \frac{1}{2}\frac{d}{dt}a(u(t), u(t)) = \mathrm{Re}(f(t), u'(t)) + \frac{1}{2}(u'(t), (A^* - A)u(t)).$$

Multiplying the both sides by t and integrating over $[0, t]$

$$\begin{aligned} &\int_0^t |u'(s)|^2 s ds + \frac{1}{2}\int_0^t s\frac{d}{ds}a(u(s), u(s))ds \\ =&\mathrm{Re}\int_0^t (f(s), u'(s)) s ds + \frac{1}{2}\int_0^t (u'(s), (A^* - A)u(s)) s ds. \end{aligned}$$

By an elementary calculation, we obtain (3.5). The proof is complete.

Set $u_0(t) = x$. Let u_1 be the solution of the following initial value problem

$$\begin{aligned} &\frac{d}{dt}u_1(t) + Au_1(t) = G(u_0)(t), \\ &u_1(0) = x. \end{aligned}$$

Since $u_0 \in C([0,h]:H)$, $G(u_0) \in L^2(0,h:V^*)$ by Proposition 3.2. Hence, by a results of J.L. Lions [7], the solution $u_1(t)$ exists.

Since $u_1(t) \in C([0,h]:H)$, $G(u_1) \in L^2(0,h:V^*)$. Hence we can define $u_2(t)$ as the solution of

$$\frac{d}{dt}u_2(t) + Au_2(t) = G(u_1)(t),$$
$$u_2(0) = x.$$

Iterating this process, one shows that there exists a sequence $\{u_n(t)\}$ such that

$$\frac{d}{dt}u_n(t) + Au_n(t) = G(u_{n-1})(t),$$
$$u_n(0) = x$$

for any $n = 1, 2, \cdots$.

To prove the convergence of $\{u_n(t)\}$, we remark the following that.

PROPOSITION 3.4. *Let $u(t)$ and $\widehat{u}(t)$ be elements of $C([0,h]:H)$, and $v(t)$, $\widehat{v}(t)$ be a solutions of the following equations :*

$$\frac{d}{dt}v(t) + Av(t) = G(u)(t)\ ,\ v(0) = x$$
$$\frac{d}{dt}\widehat{v}(t) + A\widehat{v}(t) = G(\widehat{u})(t)\ ,\ \widehat{v}(0) = x,$$

then the following inequality holds :

$$|v(t) - \widehat{v}(t)| \le (|R(0)| + V(R:0,t)) \int_0^t |u(s) - \widehat{u}(s)|ds. \tag{3.7}$$

Proof. Since

$$\frac{d}{dt}(v(t) - \widehat{v}(t)) + A(v(t) - \widehat{v}(t)) = G(u)(t) - G(\widehat{u})(t).$$

Taking the inner product of both sides and $(v(t) - \widehat{v}(t))$, we obtain

$$\frac{1}{2}\frac{d}{dt}|v(t) - \widehat{v}(t)|^2 \le |G(u)(t) - G(\widehat{u})(t)||v(t) - \widehat{v}(t)|.$$

We integrate this inequality from 0 to t, obtaining

$$\begin{aligned}&\frac{1}{2}|v(t)-\widehat{v}(t)|^2\\ \le&\frac{1}{2}|v(0)-\widehat{v}(0)|^2+\int_0^t|G(u)(s)-G(\widehat{u})(s)||v(s)-\widehat{v}(s)|ds.\end{aligned}$$

By the Grown's type Lemma of [4], we have

$$|v(t)-\widehat{v}(t)|\le\int_0^t|G(u)(s)-G(\widehat{u})(s)|ds.$$

Note that $G(u)$ and $G(\widehat{u})$ themselves do not belong to $L^1(0,h:H)$, but their difference does. By the definition of $G(u)(t)$, we obtain

$$G(u)(s)-G(\widehat{u})(s)=-R(0)(u(s)-\widehat{u}(s)-(\dot{R}*(u-\widehat{u}))(s).$$

Hence

$$|v(t)-\widehat{v}(t)|\le R(0)\int_0^t|u(s)-\widehat{u}(s)|ds+\int_0^t|(\dot{R}*(u-\widehat{u}))(s)|ds.$$

By the elementary calculation, we obtain (3.7). Applying (3.7) to u_n, u_{n-1} in place of u, $\widehat{u}$

$$|u_{n+1}(t)-u_n(t)|\le(|R(0)|+V(R:0,t))\int_0^t|u_n(s)-u_{n-1}(s)|ds.$$

If $0\le t\le h$ then $V(R:0,t)\le V(R:0,h)$. Hence, putting

$$C_0=|R(0)|+V(R:0,h),$$

we have

$$|u_{n+1}(t)-u_n(t)|\le C_0\int_0^t|u_n(s)-u_{n-1}(s)|ds. \tag{3.8}$$

Iterating (3.8) one shows by the induction the following that

$$\begin{aligned}|u_{n+1}(t)-u_n(t)|&\le C_0^n\int_0^t\frac{(t-\tau)^{n-1}}{(n-1)!}|u_1(\tau)-u_0(\tau)|d\tau\\ &\le\frac{(C_0h)^n}{n!}\max_{0\le\tau\le h}|u_1(\tau)-u_0(\tau)|.\end{aligned}$$

By the above argument, $\{u_n(t)\}$ converges uniformly in $C([0,h] : H)$.

Put $u(t) = \lim_{n\to\infty} u_n(t)$ using (3.4) , (3.5) to $\{u_n(t)\}$, we have the following that

$$C\int_0^t \|u_{n+1}(s)\|^2 ds \le |x|^2 + \frac{1}{c}\int_0^t \|G(u_n)(s)\|_*^2 ds.$$

$$\int_0^t |u'_{n+1}(s)|^2 s ds \le (1+\frac{C^2}{2c}t)|x|^2 + \frac{1}{c}(1+\frac{C^2}{2c}t)\int_0^t \|G(u_n)(s)\|_*^2 ds$$
$$+2\int_0^t |G(u_n)(s)|^2 s ds$$

As is easily seen the right hand sides of the above inequalities are bounded. Hence, we have that u and u' belong to $L^2(0,h : V)$ and $L^2(0,h : H, tdt)$, respectively, and u satisfies (1.4) and (1.2). Thus u is a solution of (1.4) and hence of (1.3). Therefore, u is a solution of (1.1).

Uniqueness follows easily from Proposition 3.4 over an interval $[0,h]$.

3.2 Construction of Solution in $[h, 2h)$

It is easy that the following :
if $t \in [h, 2h]$, then it follows that $-h \le t-2h < 0$, hence, the initial condition is $u(t-2h) = y(t-2h)$.

One obtains

$$\int_{-h}^0 a(-s)A_2u(t+s)ds = \int_{t-h}^t a(t-s)A_2u(s)ds$$
$$= \int_{t-h}^h a(t-s)A_2u(s)ds + \int_h^t a(t-s)A_2u(s)ds.$$

We put

$$f(t) = \int_{t-h}^h a(t-s)A_2u(s)ds + \int_h^t a(t-s)A_2u(s)ds.$$

The function $f(t)$ is satisfied the assumption of [5] over an interval $[h, 2h)$.

PROPOSITION 3.5. *Let x and y satisfy (2.3) and (2.4) over an interval* $[h, 2h)$. *Then the function* $f(t) \in L^2(0, h : V^*) \cap L^2(0, h : H, (t-h)dt)$ *exists in* H.

Proof. The proof of this Proposition is the same as that of Proposition 3.1 Hence, we obtain that

$$\int_{t-h}^{h} a(t-s)A_2 u(s)ds$$

is bounded in H.
By $u(t) \in L^2(0, \hbar : D(A), tdt)$ it satisfies that the following

$$f \in L^2(0, h : H, (t-h)dt).$$

In view of $\int_{+0}^{h} Au(t)dt \in H$, we obtain that $\int_{h+0}^{2h} f(t)dt$ belongs to H.
Hence, the solution of (1.3) exists in $[h, 2h)$ satisfying the initial condition $u(h) = u(h-0)$, i.e., $u(t) \in L^2(h, 2h : V) \cap L^2(h, 2h : H, (t-h)dt)$, and $\int_{h+0}^{2h} Au(t)$ exists in H.

The proof of the main theorem is almost the same as that of Section 3.

Iterating this process, one shows that there exists a solution for any $[0, T]$.

Appendix

We give an example of H, V, f such that

(A.1) $f \in L^2(0, \pi; V^*) \cap L^2(0, \pi; H, tdt)$,

(A.2) $\int_{+0}^{\pi} f(t)dt$ exist in H,

(A.3) $\int_{+0}^{\pi} |f(t)|dt = \infty$.

Let A be the operator associated with the inner product $((\cdot,\cdot))$ of V :

$$a(u, v) = ((u, v)), \quad \forall \ u, v \in V.$$

Then, the realization of A in H is positive definite and self-adjoint. For $u_0 \in H$ set $u(t) = e^{-tA}u_0$. Then it is easy to see that

$$f(t) = u'(t) = -Ae^{-tA}u_0 \tag{a.1}$$

satisfies (A.1) and (A.2).

It remains to choose H, V, u_0 so that the function $f(t)$ defined by (a.1) satisfies (A.3).

Let $H = L^2(0,\pi), V = H_0^1(0,\pi)$. Then

$$((u,v)) = \int_0^\pi \frac{du}{dx} \cdot \frac{\overline{dv}}{dx} dx \tag{a.2}$$

is an inner product in $H_0^1(0,\pi)$. The realization in $L^2(0,\pi)$ of the operator associated with (a.2) is

$$D(A) = \{u \in L^2(0,\pi) : u(0) = u(\pi) = 0\},$$

$$Au = -\Delta u \quad \text{for} \quad u \in D(A).$$

Denote the eigenvalue of A by n^2, $n = 1, 2, \cdots$, and the corresponding orthonormal set of eigenfunctions by $\varphi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$.

We use the following elementary fact :

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \begin{cases} < \infty, & \text{if } p > 1 \\ = \infty, & \text{if } p \leq 1. \end{cases} \tag{a.3}$$

Let

$$u_0 = \sum_{n=1}^{\infty} \frac{\varphi_n}{(n+1)^{1/2}(\ln(n+1))^{2/3}},$$

that is, u_0 satisfies

$$(u_0, \varphi_n) = \frac{1}{(n+1)^{1/2}(\ln(n+1))^{2/3}}.$$

Put $u(t) = e^{-tA}u_0$. Then,

$$u'(t) = -Ae^{-tA}u_0 = -\sum_{n=0}^{\infty} (u_0, \varphi_n) n^2 e^{-n^2 t} \varphi_n.$$

Since (u_0, φ_n) is a decreasing sequence

$$
\text{(a.4)} \qquad \begin{aligned}
|u'(t)|^2 &= \sum_{n=1}^{\infty} (u_0, \varphi_n)^2 n^4 e^{-2n^2 t} \\
&\geq \sum_{n=1}^{[\sqrt{\frac{1}{t}}]} n^2 e^{-2n^2 t} (u_0, \varphi_{[\sqrt{\frac{1}{t}}]})^2
\end{aligned}
$$

where [ ] is Gauss's notation. Noting that $x^4 e^{-2x^2 t}$ is an increasing function of x in the interval $[0, \sqrt{\frac{1}{t}}]$, we get

$$
\text{(a.5)} \qquad \begin{aligned}
\int_0^{\sqrt{\frac{1}{t}}} x^4 e^{-2x^2 t} dx &= \sum_{n=1}^{[\sqrt{\frac{1}{t}}]} \int_{n-1}^{n} x^4 e^{-2x^2 t} dx + \int_{[\sqrt{\frac{1}{t}}]}^{\sqrt{\frac{1}{t}}} x^4 e^{-2x^2 t} dx \\
&\leq \sum_{n=1}^{\sqrt{\frac{1}{t}}} n^4 e^{-2n^2 t} + \frac{e^{-2}}{t^2}
\end{aligned}
$$

On the other hand, by the change of the variable $x^2 t = y$, we obtain

$$
\text{(a.6)} \qquad \begin{aligned}
\int_0^{\sqrt{\frac{1}{t}}} x^4 \; e^{-2x^2 t} dx &= \int_0^1 \frac{y^2}{t^2} \; e^{-2y} \frac{1}{2\sqrt{t}} \; y^{-\frac{1}{2}} dy \\
&= \frac{t^{-\frac{5}{2}}}{2} \int_0^1 y^{\frac{3}{2}} \; e^{-2y} dy.
\end{aligned}
$$

Combining (a.5) , (a.6) we get

$$
\text{(a.7)} \qquad |u'(t)|^2 \geq \frac{t^{-\frac{5}{2}}}{4} \int_0^1 y^{\frac{3}{2}} \; e^{-2y} dy (u_0, \varphi_{[\sqrt{\frac{1}{t}}]}).
$$

Set $a = \left(\frac{e^2}{4} \int_0^1 y^{\frac{3}{2}} \; e^{-2y} dy \right)^2$, it follows from (a.7)

$$
|u'(t)| \geq c_0 \cdot t^{-\frac{5}{4}} (u_0, \varphi_{[\sqrt{\frac{1}{t}}]})
$$

for some positive constant c_0 and $0 \leq t \leq a$.

Hence, with the aid of the change of the variable $t = s^{-2}$ we have

$$\begin{aligned}\int_0^a |u'(t)|dt &\geq c_0 \int_0^a t^{-\frac{5}{4}}(u_0, \varphi_{[\sqrt{\frac{1}{t}}]})dt \\ &\geq c_0 \int_N^\infty s^{\frac{5}{2}}(u_0, \varphi_{[s]})2s^{-2}ds\end{aligned}$$

where $N = [a^{-\frac{1}{2}}]$. As is easily seen

$$\begin{aligned}\int_N^\infty s^{-\frac{1}{2}}(u_0, \varphi_{[s]})ds &= \sum_{n=N}^\infty \int_n^{n+1} s^{-\frac{1}{2}}(u_0, \varphi_{[s]})ds \\ &\geq \sum_{n=N}^\infty (n+1)^{-\frac{1}{2}}(u_0, \varphi_n) \\ &= \sum_{n=N}^\infty \frac{1}{(n+1)(\ln(n+1))^{2/3}} \\ &= \sum_{n=N+1}^\infty \frac{1}{n(\ln n)^{2/3}} = \infty.\end{aligned}$$

Thus we conclude

$$\int_0^a |f(t)|dt = \int_0^a |u'(t)|dt = \infty.$$

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Department of Mathematics,
Dong-A University,
Pusan 604-714, Korea